# Formal Logic

## Lecture 3: The Semantics of Propositional Logic (Part II)

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# Doing Logic with Truth-Tables

# Truth-table construction: Rows

- Recall that truth-tables give us the possibility to check all the possible combinations of truth values for sentences.

- The first thing we need in order to construct such a multi-line truth-table is the number *n* of distinct sentence letters.

- This number determines the number of rows, given by $2^n$, that cover all the possible combinations of truth-values.

| **Sentence** | **Number of sentence letters: *n*** | **Rows: $2^n$** |
|---|---|---|
| C | 1 | 2 |
| C&¬C | 1 | 2 |
| C&(A→D) | 3 | 8 |
| (G∨A)&(C→B) | 4 | 16 |

# Truth-table construction: Variables and sentences

- Sentence letters (a.k.a. variables) go on the left (in alphabetical order). The sentence(s) to be evaluated goes/go on the right.

$n = 2$, $2^n = 4$ rows

| A | B | $\neg A \rightarrow B$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

$n = 3$, $2^n = 8$ rows

| D | H | W | (W ∨ D) & H |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

# Truth-table construction: Variable truth-values

- Sentence letters (a.k.a. variables) go on the left (in alphabetical order). The sentence(s) to be evaluated goes/go on the right.

$n = 2$, $2^n = 4$ rows

| A | B | $\neg A \rightarrow B$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

- We then insert all the combos of T/F under the individual letter columns.

$n = 3$, $2^n = 8$ rows

| D | H | W | (W ∨ D) & H |
|---|---|---|---|
| T | T | T | |
| T | T | F | |
| T | F | T | |
| T | F | F | |
| F | T | T | |
| F | T | F | |
| F | F | T | |
| F | F | F | |

# Truth-table construction: Sentence truth-values

- For any given sentence, we start by inserting the truth-values in the columns for all the atomic sentences (i.e. the letters).

| A | B | ¬A | → | B |
|---|---|---|---|---|
| T | T | T | | T |
| T | F | T | | F |
| F | T | F | | T |
| F | F | F | | F |

# Truth-table construction: Sentence truth-values (2)

- For any given sentence, we start by inserting the truth-values in the columns for all the atomic sentences (i.e. the letters).
- We then insert the truth-values in the columns of any negations that bind with atomic sentences.

| A | B | ¬A | → | B |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

# Truth-table construction: Sentence truth-values (3)

- For any given sentence, we start by inserting the truth-values in the columns for all the atomic sentences (i.e. the letters).
- We then insert the truth-values in the columns of any negations that bind with atomic sentences.
- This is followed by the truth-values of binary connectives.

| A | B | ¬A | → | B |
|---|---|---|---|---|
| T | T | F | **T** | T |
| T | F | F | **T** | F |
| F | T | T | **T** | T |
| F | F | T | **F** | F |

# Truth-table construction: Main connective

- For any given sentence, we start by inserting the truth-values in the columns for all the atomic sentences (i.e. the letters).
- We then insert the truth-values in the columns of any negations that bind with atomic sentences.
- This is followed by the truth-values of binary connectives.

| A | B | ¬A | → | B |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | F |

- The last connective fixes the truth-values of the sentence.
- A.k.a.: main connective/column.

# Truth-table construction: Main connective (2)

- For any given sentence, we start by inserting the truth-values in the columns for all the atomic sentences (i.e. the letters).
- We then insert the truth-values in the columns of any negations that bind with atomic sentences.
- This is followed by the truth-values of binary connectives.

| A | B | ¬A → B |
|---|---|---|
| T | T | F T T T |
| T | F | F T T F |
| F | T | T F T T |
| F | F | T F F F |

- The last connective fixes the truth-values of the sentence.
- A.k.a.: main connective/column.

**NB**: Indicated by a downward arrow in the app / Logic Book.

# Truth-table construction: Brackets

- We evaluate the innermost content first (following the previous rules) and work our way to the outermost brackets.

| A | B | ¬(A → B) | |
|---|---|---|---|
| T | T | T | T |
| T | F | T | F |
| F | T | F | T |
| F | F | F | F |

| C | D | ¬(C → D) v (D → C) | | | |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | T | F | F | T |
| F | T | F | T | T | F |
| F | F | F | F | F | F |

# Truth-table construction: Brackets (2)

- We evaluate the innermost content first (following the previous rules) and work our way to the outermost brackets.

| A | B | ¬(A → B) |
|---|---|---|
| | | T T T |
| T | T | T **T** T |
| T | F | T **F** F |
| F | T | F **T** T |
| F | F | F **T** F |

| C | D | ¬(C → D) v (D → C) |
|---|---|---|
| | | T T T T T T |
| T | T | T **T** T T **T** T |
| T | F | T **F** F F **T** T |
| F | T | F **T** T T **F** F |
| F | F | F **T** F F **T** F |

# Truth-table construction: Brackets (3)

- We evaluate the innermost content first (following the previous rules) and work our way to the outermost brackets.

| A | B | ¬(A → B) |
|---|---|---|
| T | T | F T T T |
| T | F | T T F F |
| F | T | F F T T |
| F | F | F F T F |

| C | D | ¬(C → D) v (D → C) |
|---|---|---|
| T | T | F T T T T T T |
| T | F | T T F F F T T |
| F | T | F F T T T F F |
| F | F | F F T F F T F |

# Truth-table construction: Brackets (4)

- We evaluate the innermost content first (following the previous rules) and work our way to the outermost brackets.

| A | B | ¬(A → B) |
|---|---|---|
| T | T | F T T T |
| T | F | T T F F |
| F | T | F F T T |
| F | F | F F T F |

| C | D | ¬(C → D) v (D → C) |
|---|---|---|
| T | T | F T T T **T** T T T |
| T | F | T T F F **T** F T T |
| F | T | F F T T **F** T F F |
| F | F | F F T F **T** F T F |

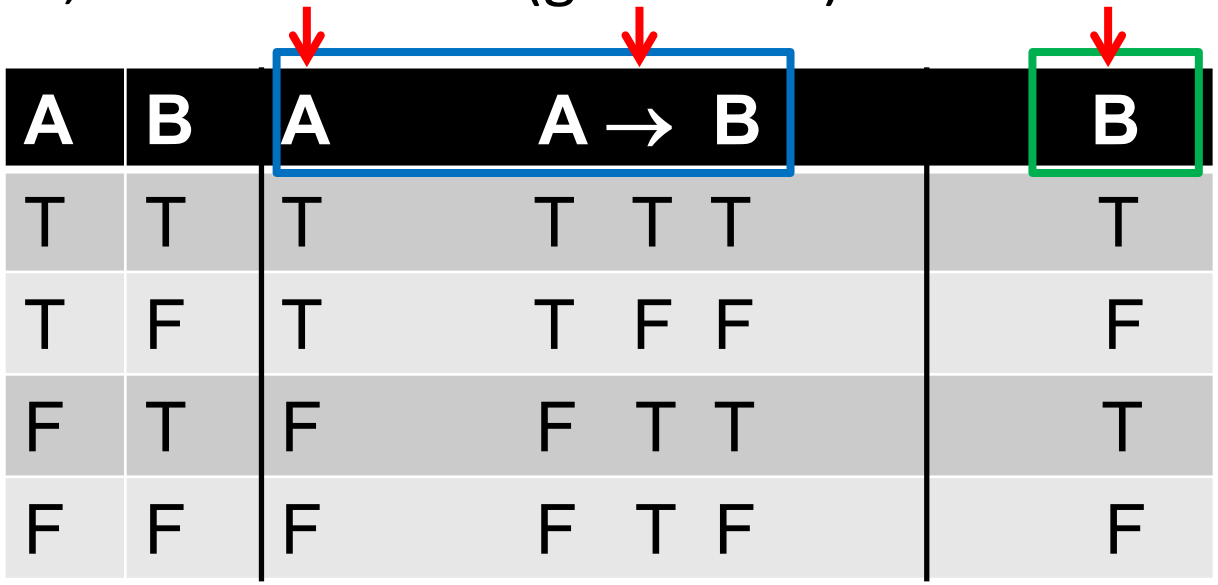
# Determining validity

- Validity, you may recall, is a property of arguments, not sentences. How do we express arguments in truth-tables?
- We list the premises (blue box) to the right of the sentence letters; the conclusion (green box) is listed to the very right.

| A | B | A | A → B | B |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

# Determining validity (2)

- Validity, you may recall, is a property of arguments, not sentences. How do we express arguments in truth-tables?
- We list the premises (blue box) to the right of the sentence letters; the conclusion (green box) is listed to the very right.

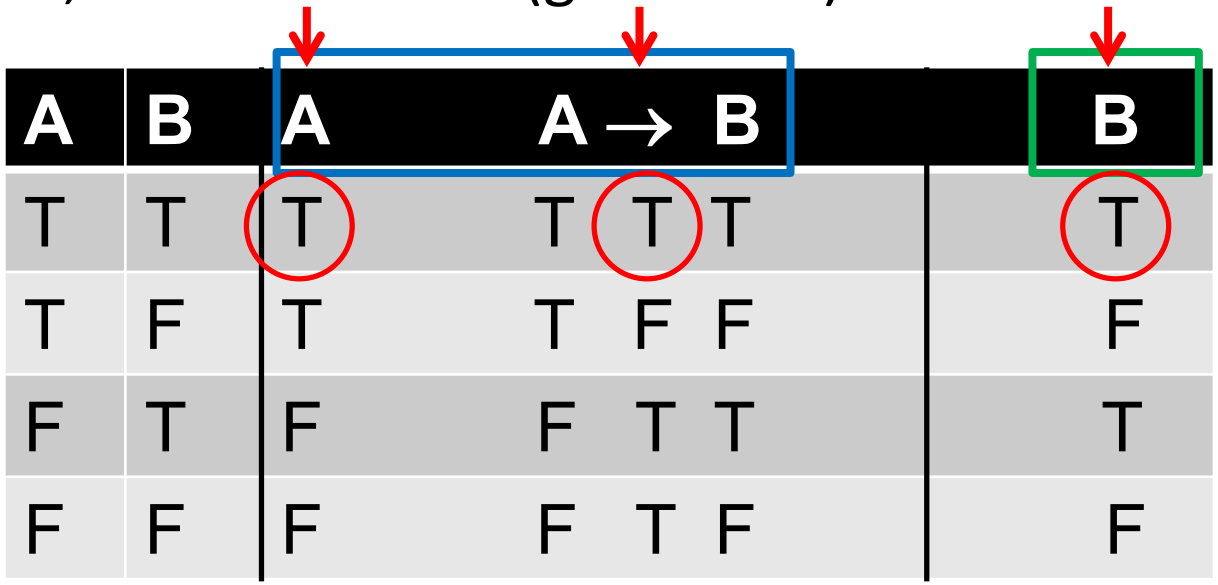


| A | B | A | A → B | B |
|---|---|---|---|---|
| T | T | T | T T T | T |
| T | F | T | T F F | F |
| F | T | F | F T T | T |
| F | F | F | F T F | F |

- **Validity**: Just in case there is no row (truth-value assignment) where the premises are true and the conclusion is false.

# Determining validity: Example 1

- Validity, you may recall, is a property of arguments, not sentences. How do we express arguments in truth-tables?
- We list the premises (blue box) to the right of the sentence letters; the conclusion (green box) is listed to the very right.



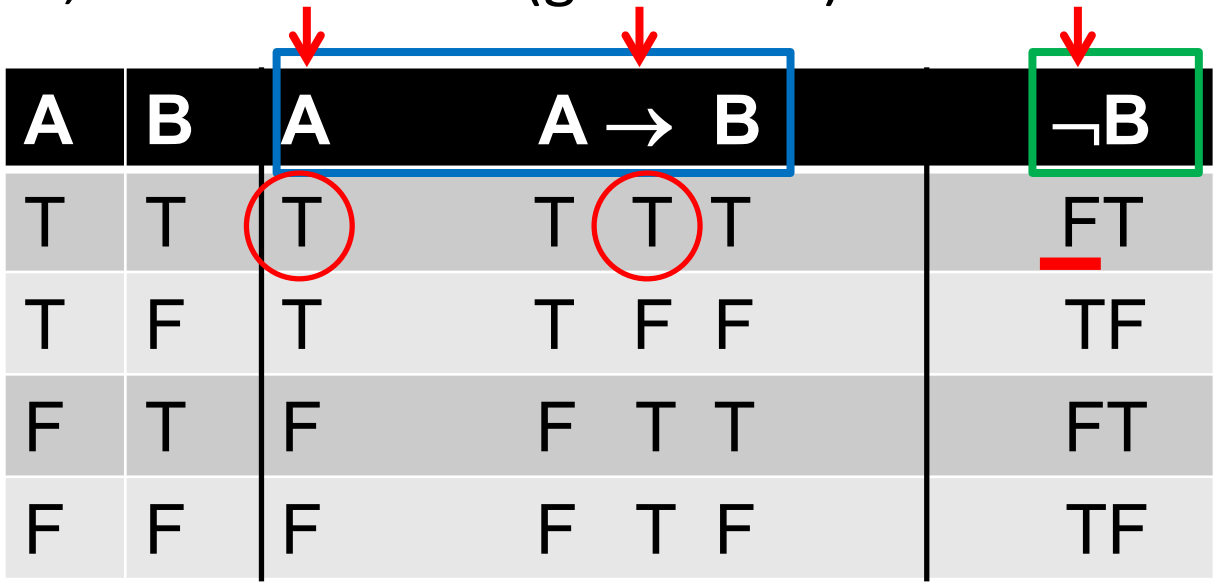
| A | B | A | A → B | | B |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | T | T | F | F |
| F | T | F | F | T | T |
| F | F | F | F | T | F |

**VALID**

- **Validity**: Just in case there is no row (truth-value assignment) where the premises are true and the conclusion is false.

# Determining validity: Example 2

- Validity, you may recall, is a property of arguments, not sentences. How do we express arguments in truth-tables?
- We list the premises (blue box) to the right of the sentence letters; the conclusion (green box) is listed to the very right.



| A | B | A | A → B | ¬B |
|---|---|---|---|---|
| T | T | T | T T T | FT |
| T | F | T | T F F | TF |
| F | T | F | F T T | FT |
| F | F | F | F T F | TF |

**INVALID**

- **Validity**: Just in case there is no row (truth-value assignment) where the premises are true and the conclusion is false.

# Semantic validity

- The notion of validity just explicated is known as:

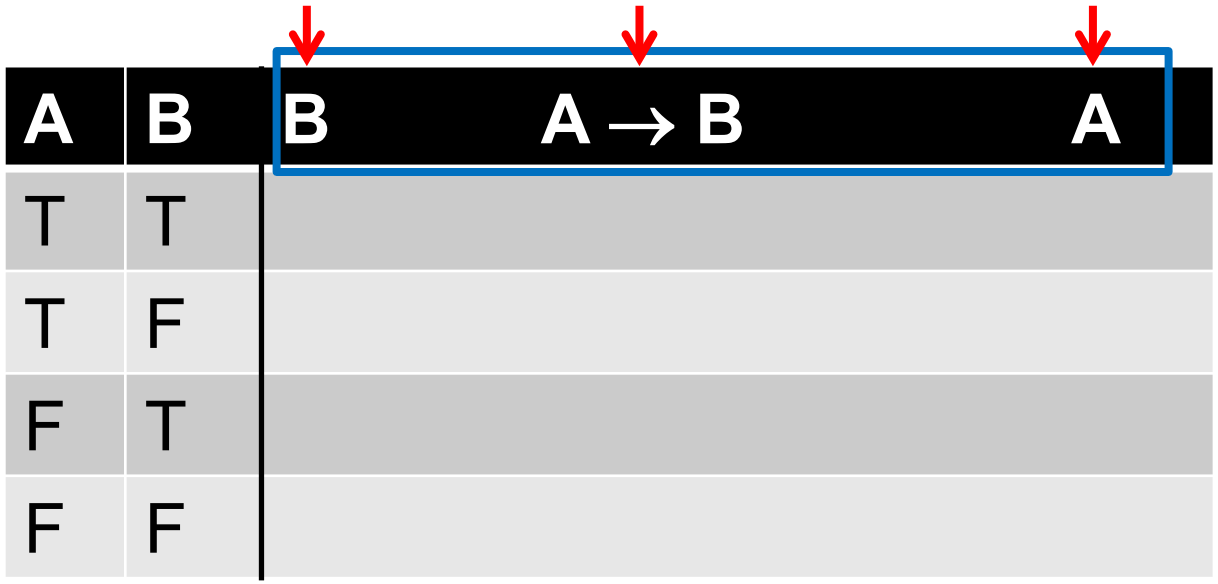
  * semantic validity
  * truth-functional validity
  * semantic entailment

- We use the symbol '⊨' to denote semantic validity:

| **Premise(s)** | **Entail** | **Conclusion** |
|---|---|---|
| P | ⊨ | P∨Q |
| {P, P→Q} | ⊨ | Q |
| {A, ¬A∨B} | ⊨ | B |
| { } | ⊨ | R∨¬R |

# Determining consistency

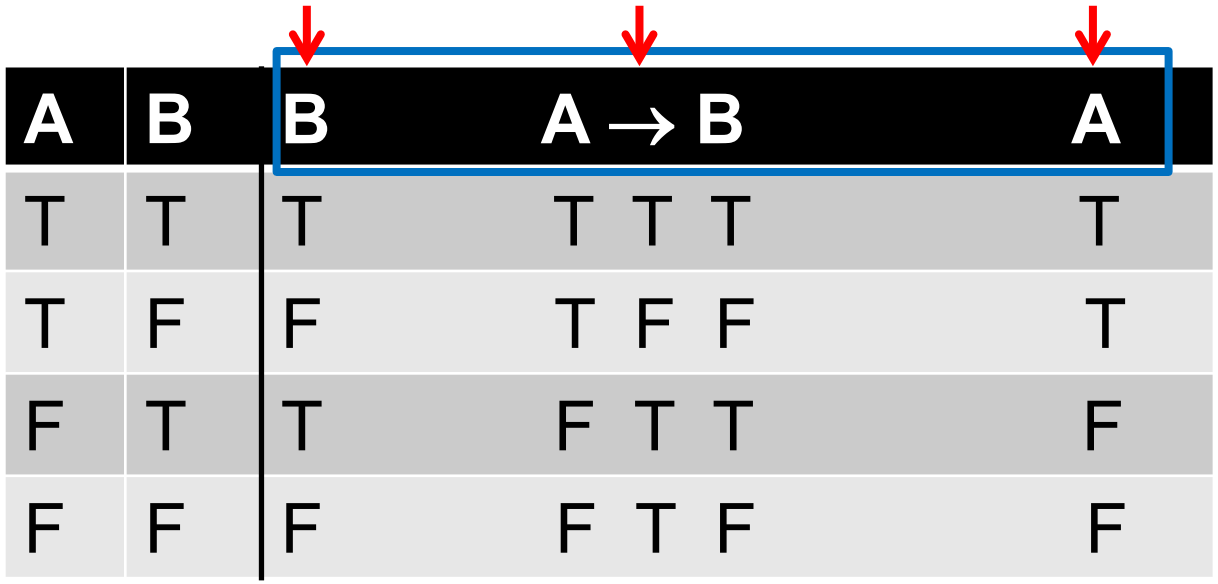
- Consistency is a property of sets of sentences. How do we express sets of sentences in truth-tables?
- We list the sentences (blue box) separately to the right of the sentence letters.



| A | B | B | $A \rightarrow B$ | A |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

# Determining consistency (2)

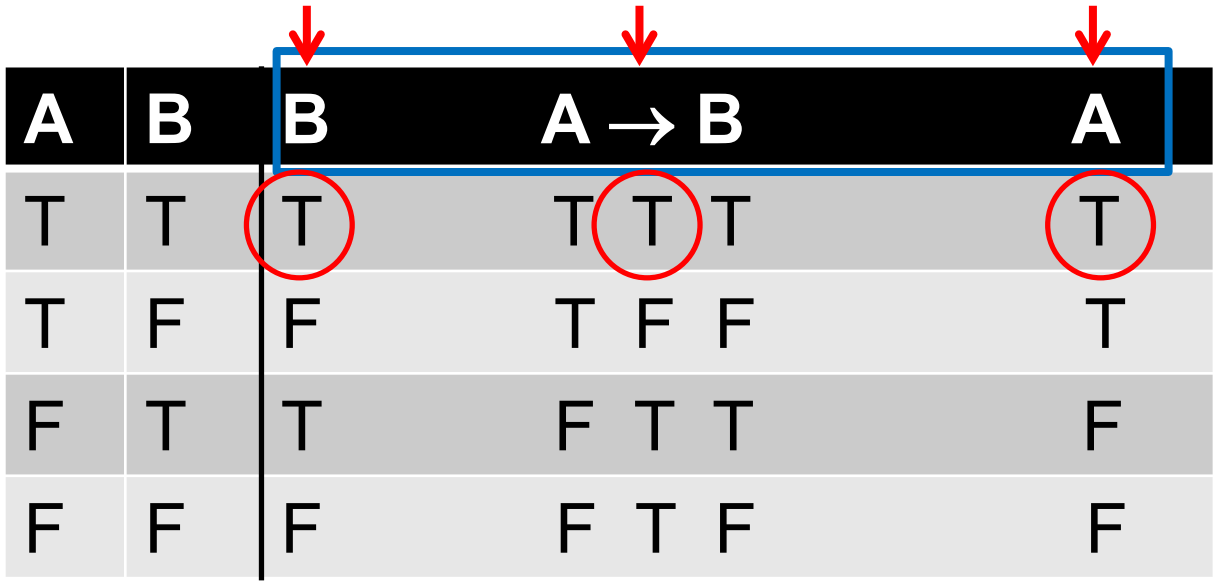
- Consistency is a property of sets of sentences. How do we express sets of sentences in truth-tables?
- We list the sentences (blue box) separately to the right of the sentence letters.



| A | B | B | A → B | | | A |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | F |
| F | F | F | F | T | F | F |

# Determining consistency: Example 1

- Consistency is a property of sets of sentences. How do we express sets of sentences in truth-tables?
- We list the sentences (blue box) separately to the right of the sentence letters.



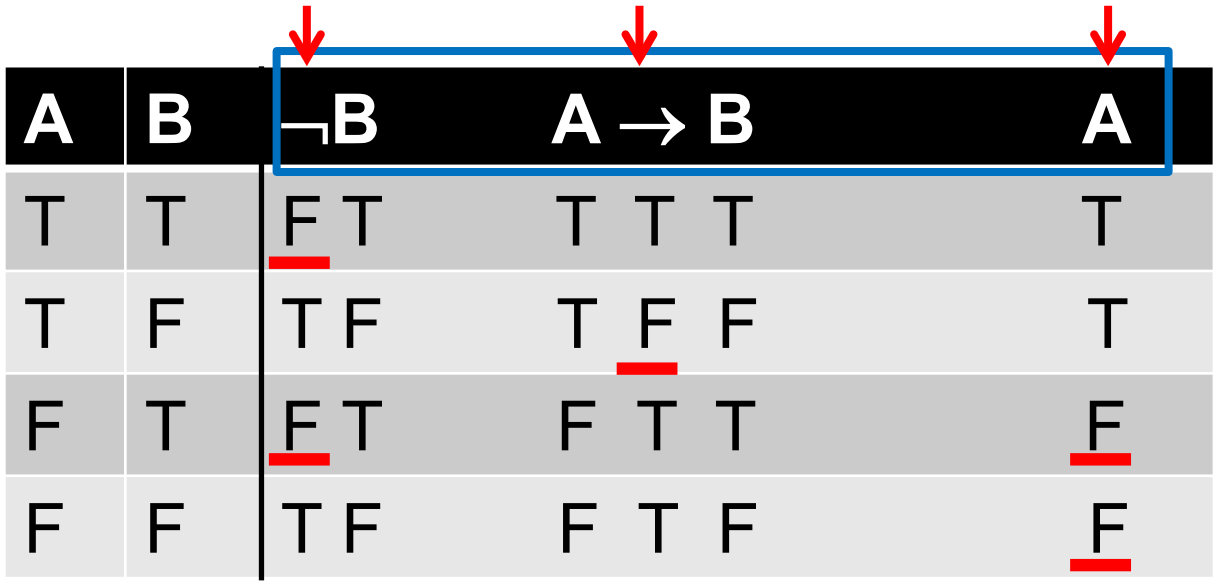
| A | B | B | A → B | | | A |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | F |
| F | F | F | F | T | F | F |

**CONSISTENT**

- **Semantic consistency**: Just in case there is at least one row (truth-value assignment) where all the sentences are true.

# Determining consistency: Example 2

- Consistency is a property of sets of sentences. How do we express sets of sentences in truth-tables?
- We list the sentences (blue box) separately to the right of the sentence letters.



| A | B | ¬B | A → B | A |
|---|---|---|---|---|
| T | T | F T | T T T | T |
| T | F | T F | T F F | T |
| F | T | F T | F T T | F |
| F | F | T F | F T F | F |

**INCONSISTENT**

- **Semantic consistency**: Just in case there is at least one row (truth-value assignment) where all the sentences are true.

# Types of sentences

- A sentence (whether atomic or complex) belongs to one of three types. The types are mutually exclusive and exhaustive.

  (1) Contingent sentences

  (2) Tautologies

  (3) Contradictions

# Determining contingent sentences

- When the main column of a truth-table yields both Ts and Fs, we say that the sentence is **contingent**.

- **NB**: Also known as contingently true or logically contingent.

| A | B | ¬(A & B) |
|---|---|---|
| T | T | F T T T |
| T | F | T T F F |
| F | T | T F F T |
| F | F | T F F F |

| A | B | A ∨ (A ↔ B) |
|---|---|---|
| T | T | T T T T T |
| T | F | T T T F F |
| F | T | F F F F T |
| F | F | F T F T F |

- A sentence is **logically contingent** if and only if it is true/false under some but not all truth-value assignments.

# Determining tautologies

- When the main column of a truth-table for a *single* sentence yields only Ts, we say that the sentence is a **tautology**.

  **NB**: Also known as logical theorems or logical truths.

| A | (A ∨ ¬A) |
|---|---|
| T | T **T** FT |
| F | F **T** TF |

| A | B | A → (A ∨ B) |
|---|---|---|
| T | T | T **T** T T T |
| T | F | T **T** T T F |
| F | T | F **T** F T T |
| F | F | F **T** F F F |

- A sentence is **logically true** if and only if it is true under all truth-value assignments.

# Determining contradictions

- When the main column of a truth-table for a *single* sentence yields only Fs, we say that the sentence is a **contradiction**.

  **NB**: Also known as absurdities or logical falsities.

| A | (A & ¬A) |
|---|---|
| T | T F FT |
| F | F F TF |

| A | B | (A & B) ↔ ¬(A & B) |
|---|---|---|
| T | T | T T T F F T T T |
| T | F | T F F F T T F F |
| F | T | F F T F T F F T |
| F | F | F F F F T F F F |

- A sentence is **logically false** if and only if it is false under all truth-value assignments.

# Relating the two notions

A sentence is a **tautology** IFF its *negation is a contradiction*.

A sentence is a **contradiction** IFF its *negation is a tautology*.

# Determining logical equivalence

- When the main columns of two sentences are identical, we say that the sentences are logically equivalent.

| A | B | A → B | ¬A ∨ B |
|---|---|---|---|
| T | T | T **T** T | FT **T** T |
| T | F | T **F** F | FT **F** F |
| F | T | F **T** T | TF **T** T |
| F | F | F **T** F | TF **T** F |

- Two sentences are **logically equivalent** if and only if they are true/false under exactly the same truth-value assignments.

# The End