Formal Logic

Lecture 4: Derivations in Propositional Logic

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Derivations in Propositional Logic

Doing logic with syntax

- In the last two lectures, we used semantic methods (particularly truth-tables) to determine:

  * Validity (and invalidity)
  * Consistency (and inconsistency)
  * Contingent sentences
  * Tautologies
  * Contradictions

- But there is also a syntactic way to determine the same properties. For that we need a derivation system: **SD**.

- SD consists of 11 inference (a.k.a. derivation) rules that specify when an L_1 proposition can be derived from another.

Syntactic validity

- A proposition P is **derivable (in SD)** from set of premises Γ iff P can be reached from Γ only by applying SD's inference rules.

- An argument with a set of premises Γ and conclusion P is **syntactically valid (in SD)** iff P is derivable in SD from Γ.

- We use the symbol '$\vdash$' to denote syntactic validity:

Premise(s)	**Derives**	**Conclusion**
P	$\vdash$	P∨Q
{P, P→Q}	$\vdash$	Q
{A, ¬A∨B}	$\vdash$	B
{ }	$\vdash$	R∨¬R

Soundness and completeness

- The holy grail for a logical system is when its syntactic and semantic validities are equally powerful.

- That is indeed the case with propositional logic as it is both sound and complete (two meta-theorems).

  Soundness: Any proposition that is syntactically derivable from a set Γ is semantically entailed by Γ.

  Completeness: Any proposition that is semantically entailed by Γ is syntactically derived from Γ.

  NB: The notion of soundness here is different than the one we saw before (i.e. That notion applies to arguments).

The eleven inference rules in SD

Name	Abbreviation
* Reiteration	R
* Conjunction Introduction	&I
* Conjunction Elimination	&E
* Disjunction Introduction	$\vee$I
* Disjunction Elimination	$\vee$E
* Conditional Introduction	$\rightarrow$I
* Conditional Elimination	$\rightarrow$E
* Bi-conditional Introduction	$\leftrightarrow$I
* Bi-conditional Elimination	$\leftrightarrow$E
* Negation Introduction	$\neg$I
* Negation Elimination	$\neg$E

NB: Conditional elimination is a.k.a. *modus ponens*.

Other inference rules

- A number of other inference rules could have been included in our logical system:

  * Modus Tollens
  * Hypothetical Syllogism
  * Disjunctive Syllogism
  * ...

- The more one includes the easier (shorter) the derivations become but also the more one needs to memorise.

- SD uses only the rules listed on the previous slide.

Replacement rules

- There are also various replacement rules that allow for shorter derivations:

  * Absorption
  * Association
  * Commutation
  * De Morgan's Laws
  * Distribution
  * Double Negation
  * Exportation
  * Idempotence
  * Transposition/Contraposition

  NB: We won't be using these in the course – they're not included in SD – but feel free to look them up.

The schematic structure of derivations

- Suppose we want to derive a proposition *Z* from a set of *n* premises. The structure of that derivation looks like this:

Derivation		**Justification for this step:**
1.	*Premise 1*	Assumption
2.	*Premise 2*	Assumption
…		
n.	*Premise n*	Assumption
n+1.	*Some Proposition*	Some rule (w/steps involved)
n+2.	*Some Proposition*	Some rule (w/steps involved)
…		
n+m.	*Proposition Z*	Some rule (w/steps involved)

NB: No step can be taken without justification, i.e. without citing some rule of inference.

Reiteration

- We begin with the simplest rule:

Reiteration:		**Natural Language Example**:
1. A	Assumption	1. David is sad.
2. A	1 R	∴ David is sad.

Conjunction: Introduction and elimination

Conjunction Introduction		**Natural Language Example:**
1. A	Assumption	1. Joe is rich.
2. B	Assumption	2. Kate is fierce.
3. A & B	1,2 &I	∴ Joe is rich and Kate is fierce.

NB: We can also derive B & A from the same premises.

Conjunction Elimination		**Natural Language Example:**
1. A & B	Assumption	1. Chris sings and Amy works hard.
2. A	1 &E	∴ Amy works hard.

NB: We can also derive B from the same premises.

Sub-derivations

- Some inference rules require the use of sub-derivations. These make additional (for argument's sake) assumptions.

- These rules specify exactly which, if any, propositions inside a sub-derivation can be exported to the main derivation.

1.	Some Premise	Assumption
2.	| Some Auxiliary	A / Specific sub-deriv. rule
3.	| Some Proposition	Some rule (w/steps)
4.	| ...	
5.	Some Proposition	Specific sub-deriv. rule (w/steps)

Indented vertical line: scope of the sub-derivation

Indented horizontal line: identifies auxiliary assumption

Conditional: Introduction and elimination

Conditional Elimination:		**Natural Language Example:**
1. $A \rightarrow B$	Assumption	If Tim wins, Jason loses.
2. A	Assumption	Tim wins.
3. B	1, 2 $\rightarrow$E	∴ Jason loses.

NB: As I already said, $\rightarrow$E is also known as 'Modus Ponens'.

Conditional Introduction:		**Natural Language Example:**
1. $A \rightarrow B$	Assumption	If Jim screams, Louise sneezes.
2. $B \rightarrow C$	Assumption	If Louise sneezes, Henry laughs.
3. | A	A / $\rightarrow$I	| Jim screams.
4. | B	1, 3 $\rightarrow$E	| Louise sneezes.
5. | C	2, 4 $\rightarrow$E	| Henry laughs.
6. $A \rightarrow C$	3-5 $\rightarrow$I	∴ If Jim screams, Henry laughs.

Disjunction: Introduction and elimination

Disjunction Introduction:		**Natural Language Example:**
1. A	Assumption	Tom is in Bern.
2. $A \vee B$	1 ∨I	∴ Tom is in Bern or in Basel.

NB: Any new disjunct is permissible, e.g. $A \vee \neg A$

Disjunction Elimination:		**Natural Language Example:**
1. $A \vee B$	Assumption	The coffee has milk or cream.
2. $A \rightarrow C$	Assumption	If it has milk, Jane is happy.
3. $B \rightarrow C$	Assumption	If it has cream, Jane is happy.
4. | A	A / ∨E	| The coffee has milk.
5. | C	2, 4 →E	| Jane is happy.
6. | B	A / ∨E	| The coffee has cream.
7. | C	3, 6 →E	| Jane is happy.
8. C	1, 4-5, 6-7 ∨E	∴ Jane is happy.

Bi-conditional: Introduction and elimination

Bi-conditional Elimination:		Natural Language Example:
1. $A \leftrightarrow B$	Assumption	Dee leaves iff Cid stays.
2. A	Assumption	Dee leaves.
3. B	1, 2 $\leftrightarrow$E	∴ Cid stays.

NB: If premise 2 was B, we could derive A w/the same rule.

Bi-conditional Introduction:		Natural Language Example:
1. $A \rightarrow B$	Assumption	If it rains, it pours.
2. $B \rightarrow A$	Assumption	If it pours, it rains.
3. | A	A / $\leftrightarrow$I	| It rains.
4. | B	1, 3 $\rightarrow$E	| It pours.
5. | B	A / $\leftrightarrow$I	| It pours.
6. | A	2, 5 $\rightarrow$E	| It rains.
7. $A \leftrightarrow B$	3-4, 5-6 $\leftrightarrow$I	∴ It rains iff it pours.

Negation introduction

Negation Introduction:

1.	$A \rightarrow B$	Assumption
2.	$\neg B$	Assumption
3.		A	A / ¬I
4.		B	1, 3 →E
5.		$\neg B$	2 R
6.	$\neg A$	3-5 ¬I

Natural Language Example:

If Ken cries, Barbie cries.

Barbie doesn’t cry.

| Ken cries.

| Barbie cries.

| Barbie doesn’t cry.

∴ Ken doesn’t cry.

Negation elimination

Negation Elimination:		**Natural Language Example:**
1. $\neg A \rightarrow \neg B$	Assumption	If it's not warm, it won't start.
2. B	Assumption	It will start.
3. $\quad \mid \neg A$	A / $\neg E$	$\quad \mid$ It's not warm.
4. $\quad \mid \neg B$	1, 3 $\rightarrow E$	$\quad \mid$ It won't start.
5. $\quad \mid B$	2 R	$\quad \mid$ It will start.
6. A	3-5 $\neg E$	$\therefore$ It's warm.

The End