Formal Logic

Lecture 5: Derivations in Propositional Logic (Part II)

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The Principle of Explosion

Disjunctive syllogism

- On the next slide, I make use of an inference rule I left out of our system of logic to make the derivation shorter.

  NB: Try to replace the steps employing it with the rules we included in our system of logic.

Disjunctive Syllogism:		**Natural Language Example:**
1. A ∨ B	Assumption	Either Ann or Joe runs the race.
2. ¬A	Assumption	It's not the case that Ann runs.
3. B	1, 2 DS	∴ Joe runs the race.

Proof

- In what follows, we prove in four steps (in addition to the premise) that anything follows from a contradiction.
- That is, we start the derivation with a contradiction (any will do) and end with a proposition (again, any will do).

1. ¬A & A	Assumption
2. A	1 &E
3. A V Z	2 VI
4. ¬A	1 &E
5. Z	3, 4 DS

NB: As we haven't interpreted A or Z, the proof is general.

Proof (in natural language)

1. Elvis is alive and Elvis is not alive.	Assumption
2. Elvis is alive.	1 &E
3. Elvis is alive or the world will end in 2021.	2 VI
4. Elvis is not alive.	1 &E
5. The world will end in 2021.	3, 4 DS

Syntactic Properties

Syntactic consistency

- A set of propositions Γ is **syntactically consistent (in SD)** iff it is not syntactically inconsistent (in SD).
- A set of propositions Γ is **syntactically inconsistent (in SD)** iff at least one proposition and its negation is derivable from Γ.

1. $A \rightarrow B$	Assumption
2. A	Assumption
3. ¬B	Assumption
4. B	1, 2 $\rightarrow$E
5. ¬B	3 R

Theorems and anti-theorems in SD

- A proposition P is a **syntactic theorem (in SD)** iff P is derivable from the empty set (i.e. from no premises).

1.	$\quad\mid$ A	A / $\rightarrow$I
2.	$\quad\mid$ A	1 R
3.	A $\rightarrow$ A	1-2 $\rightarrow$I

- A proposition P is a **syntactic anti-theorem (in SD)** iff ¬P is a theorem (in SD).

  NB: Needless to say, these syntactic notions correspond to the semantic notions of tautology and contradiction respectively.

Syntactic contingency in SD

- A proposition P is **syntactically contingent (in SD)** iff neither P nor ¬P is a theorem (in SD).

  NB: In *The Logic Book*, these are called 'syntactically undetermined' propositions.

- Take proposition $A \rightarrow B$. Neither $A \rightarrow B$ nor $\neg(A \rightarrow B)$ can be derived from the empty set. It is thus syntactically contingent.

Syntactic equivalence in SD

- Propositions P, Q are **syntactically equivalent (in SD)** iff P is derivable (in SD) from Q and Q is derivable (in SD) from P.

1.	$A \leftrightarrow B$	Assumption
2.	| B	A / $\leftrightarrow$I
3.	| A	1, 2 $\leftrightarrow$E
4.	| A	A / $\leftrightarrow$I
5.	| B	1, 4 $\leftrightarrow$E
6.	$B \leftrightarrow A$	2-3, 4-5 $\leftrightarrow$I

1.	$B \leftrightarrow A$	Assumption
2.	| A	A / $\leftrightarrow$I
3.	| B	1, 2 $\leftrightarrow$E
4.	| B	A / $\leftrightarrow$I
5.	| A	1, 4 $\leftrightarrow$E
6.	$A \leftrightarrow B$	2-3, 4-5 $\leftrightarrow$I

Some Exercises

The Logic Book: Section 5.1.1E (Exercise 1)

Instructions: Complete the following derivations by entering justifications for the derived sentences.

The Logic Book: Section 5.1.1E

(1a) Derive: A & B

1. A	Assumption
2. A → B	Assumption
3. B	1, 2 →E
4. A & B	1, 3 &I

(1b) Derive: ¬C

1. A → (B & ¬C)	Assumption
2. A & B	Assumption
3. A	2 &E
4. B & ¬C	1, 3 →E
5. ¬C	4 &E

The Logic Book: Section 5.1.1E

(1c) Derive: ¬(A ↔ ¬B)

1. ¬(A ↔ ¬B) ↔ (¬C V ¬D)	Assumption
2 A → (¬D & C)	Assumption
3 D & A	Assumption
4 A	3 &E
5 ¬D & C	2, 4 →E
6 ¬D	5 &E
7 ¬C V ¬D	6 VI
8 ¬(A ↔ ¬B)	1, 7 ↔E

The Logic Book: Section 5.1.1E (Exercise 2)

Instructions: Complete the following derivations.

(2a) Derive: D & ¬B

1. A & ¬B	Assumption
2. (A ∨ ¬C) → D	Assumption
3. A	1 &E
4. A ∨ ¬C	3 ∨I
5. D	2, 4 →E
6. ¬B	1 &E
7. D & ¬B	5, 6 &I

The Logic Book: Section 5.1.1E

(2b) Derive: F & ¬H

1. F ↔ ¬G	Assumption
2. D → ¬G	Assumption
3. ¬H & D	Assumption
4. D	3 &E
5. ¬G	2, 4 →E
6. F	1, 5 ↔E
7. ¬H	3 &E
8. F & ¬H	6, 7 &I

(2c) Derive: ¬D ∨ E

1. A & ¬B	Assumption
2 ¬B ↔ (A ↔ ¬D)	Assumption
3. ¬B	1 &E
4. A ↔ ¬D	2, 3 ↔E
5. A	1 &E
6. ¬D	4, 5 ↔E
7. ¬D ∨ E	6 ∨I

Instructions: Complete the following derivations by entering the appropriate justifications.

The Logic Book: Section 5.1.2E

(1a) Derive: $(A \rightarrow B) \,\&\, (A \rightarrow \neg D)$

1.	$A \rightarrow (B \,\&\, \neg D)$	Assumption
2.	| A	A / $\rightarrow$I
3.	| $B \,\&\, \neg D$	1, 2 $\rightarrow$E
4.	| B	3 &E
5.	$A \rightarrow B$	2-4 $\rightarrow$I
6.	| A	A / $\rightarrow$I
7.	| $B \,\&\, \neg D$	1, 6 $\rightarrow$E
8.	| $\neg D$	7 &E
9.	$A \rightarrow \neg D$	6-8 $\rightarrow$I
10.	$(A \rightarrow B) \,\&\, (A \rightarrow \neg D)$	5, 9 &I

(1b) Derive: $A \rightarrow [B \rightarrow (C \vee D)]$

1.	$(A \& B) \rightarrow C$	Assumption
2.	$\quad A$	A / $\rightarrow$I
3.	$\quad\quad B$	A / $\rightarrow$I
4.	$\quad\quad A \& B$	2, 3 &I
5.	$\quad\quad C$	1, 4 $\rightarrow$E
6.	$\quad\quad C \vee D$	5 $\vee$I
7.	$\quad B \rightarrow (C \vee D)$	3-6 $\rightarrow$I
8.	$A \rightarrow [B \rightarrow (C \vee D)]$	2-7 $\rightarrow$I

The Logic Book: Section 5.1.2E (Exercise 2)

Instructions: Complete the following derivations.

(2a) Derive: $A \leftrightarrow B$

1.	A	Assumption
2.	B	Assumption
3.	$\quad \mid A$	A / $\leftrightarrow$I
4.	$\quad \mid B$	2 R
5.	$\quad \mid B$	A / $\leftrightarrow$I
6.	$\quad \mid A$	1 R
7.	$A \leftrightarrow B$	3-4, 5-6 $\leftrightarrow$I

(2b) Derive: ¬B

1.	B → ¬B	Assumption
2.	| B	A / ¬I
3.	| ¬B	1, 2 →E
4.	| B	2 R
5.	¬B	2-4 ¬I

The Logic Book: Section 5.1.2E

(2c) Derive: A

1. ¬¬A		Assumption
2.	¬A	A / ¬E
3.	¬¬A	1 R
4.	¬A	2 R
5. A		2-4 ¬E

Derivation Rules of Thumb

Reasoning backwards

- If the ultimate goal sentence is atomic, the final step in the derivation will see the application of an elimination rule.

  NB: In special cases like establishing inconsistency that final rule may simply be reiteration.

- If it's a compound sentence, the final step in the derivation may involve either an introduction or an elimination rule.

  NB: There are generally multiple ways to derive the same conclusion from the same premises.

- We can apply the same reasoning to the 2nd-to-last, 3rd-to-last, ... sentence until we close the premises-conclusion gap.

To subderive or not to subderive

- Another rule of thumb is to ask whether the ultimate goal sentence is a constituent part of a complex sentence premise.

- Such parts can take the form of conjuncts, disjuncts, consequents and either side of a biconditional.

- If so, the need for a subderivation for *that* sentence is unlikely. What is needed instead is an elimination rule.

  NB: Disjunction elimination is the exception here as it's both an elimination rule and employs subderivations.

- Otherwise, it is advisable to consider a subderivation.

The End