Formal Logic

Lecture 5: Derivations in Propositional Logic (Part II)

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The Principle of Explosion
On the next slide, I make use of an inference rule I left out of our system of logic to make the derivation shorter.

**NB:** Try to replace the steps employing it with the rules we included in our system of logic.

### Disjunctive Syllogism:

1. A ∨ B  Assumption Either Ann or Joe runs the race.
2. ¬A  Assumption It’s not the case that Ann runs.
3. B  1, 2 DS  \[\therefore\] Joe runs the race.

### Natural Language Example:

Either Ann or Joe runs the race.
It’s not the case that Ann runs.
\[\therefore\] Joe runs the race.
Proof

• In what follows, we prove in four steps (in addition to the premise) that anything follows from a contradiction.

• That is, we start the derivation with a contradiction (any will do) and end with a proposition (again, any will do).

1. \( \neg A \land A \) Assumption

2. \( A \) 1 &E

3. \( A \lor Z \) 2 V1

4. \( \neg A \) 1 &E

5. \( Z \) 3, 4 DS

**NB:** As we haven’t interpreted A or Z, the proof is general.
Proof (in natural language)

1. Elvis is alive and Elvis is not alive.  
   Assumption
2. Elvis is alive.  
   1 &E
3. Elvis is alive or the world will end in 2021.  
   2 V I
4. Elvis is not alive.  
   1 &E
5. The world will end in 2021.  
   3, 4 DS
Syntactic Properties
Syntactic consistency

• A set of propositions $\Gamma$ is syntactically consistent (in SD) iff it is not syntactically inconsistent (in SD).

• A set of propositions $\Gamma$ is syntactically inconsistent (in SD) iff at least one proposition and its negation is derivable from $\Gamma$.

1. $A \rightarrow B$ Assumption
2. $A$ Assumption
3. $\neg B$ Assumption
4. $B$ 1, 2 $\rightarrow E$
5. $\neg B$ 3 R
Theorems and anti-theorems in SD

• A proposition P is a **syntactic theorem (in SD)** iff P is derivable from the empty set (i.e. from no premises).

  1. \[ \frac{}{A \rightarrow I} \]
  2. \[ \frac{}{A \rightarrow 1 R} \]
  3. \[ A ightarrow A \rightarrow I \]

• A proposition P is a **syntactic anti-theorem (in SD)** iff \(\neg P\) is a theorem (in SD).

**NB**: Needless to say, these syntactic notions correspond to the semantic notions of tautology and contradiction respectively.
A proposition $P$ is **syntactically contingent (in SD)** iff neither $P$ nor $\neg P$ is a theorem (in SD).

**NB:** In *The Logic Book*, these are called ‘syntactically undetermined’ propositions.

- Take proposition $A \rightarrow B$. Neither $A \rightarrow B$ nor $\neg (A \rightarrow B)$ can be derived from the empty set. It is thus syntactically contingent.
Propositions $P, Q$ are **syntactically equivalent (in SD)** iff $P$ is derivable (in SD) from $Q$ and $Q$ is derivable (in SD) from $P$.

<table>
<thead>
<tr>
<th>1. $A \leftrightarrow B$</th>
<th>Assumption</th>
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</thead>
<tbody>
<tr>
<td>2. $\underline{B}$</td>
<td>$A / \leftrightarrow I$</td>
</tr>
<tr>
<td>3. $\underline{A}$</td>
<td>$1, 2 \leftrightarrow E$</td>
</tr>
<tr>
<td>4. $\underline{A}$</td>
<td>$A / \leftrightarrow I$</td>
</tr>
<tr>
<td>5. $\underline{B}$</td>
<td>$1, 4 \leftrightarrow E$</td>
</tr>
<tr>
<td>6. $B \leftrightarrow A$</td>
<td>$2-3, 4-5 \leftrightarrow I$</td>
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Some Exercises
Instructions: Complete the following derivations by entering justifications for the derived sentences.
(1a) Derive: A & B

1. A => Assumption
2. A & B => Assumption
3. B 1, 2 \rightarrow E
4. A & B 1, 3 &I

(1b) Derive: \neg C

1. A \rightarrow (B & \neg C) => Assumption
2. A & B => Assumption
3. A 2 &E
4. B & \neg C 1, 3 \rightarrow E
5. \neg C 4 &E
(1c) Derive: \(\neg(A \leftrightarrow \neg B)\)

1. \(\neg(A \leftrightarrow \neg B) \leftrightarrow (\neg C \lor \neg D)\) Assumption
2. \(A \rightarrow (\neg D \& C)\) Assumption
3. \(D \& A\) Assumption
4. \(A\) 3 &E
5. \(\neg D \& C\) 2, 4 \(\rightarrow\)E
6. \(\neg D\) 5 &E
7. \(\neg C \lor \neg D\) 6 \(\lor\)I
8. \(\neg(A \leftrightarrow \neg B)\) 1, 7 \(\leftrightarrow\)E
Instructions: Complete the following derivations.
(2a) Derive: D & ¬B

1. A & ¬B Assumption
2. (A ∨ ¬C) → D Assumption
3. A 1 &E
4. A ∨ ¬C 3 ∨I
5. D 2, 4 →E
6. ¬B 1 &E
7. D & ¬B 5, 6 &I
(2b) Derive: \( F \land \neg H \)

1. \( F \leftrightarrow \neg G \)  
   Assumption
2. \( D \to \neg G \)  
   Assumption
3. \( \neg H \land D \)  
   Assumption
4. \( D \)  
   3 &E
5. \( \neg G \)  
   2, 4 \to E
6. \( F \)  
   1, 5 \leftrightarrow E
7. \( \neg H \)  
   3 &E
8. \( F \land \neg H \)  
   6, 7 \&I
(2c) Derive: $\neg D \lor E$

1. $A \land \neg B$ Assumption
2. $\neg B \leftrightarrow (A \leftrightarrow \neg D)$ Assumption
3. $\neg B$ 1 &E
4. $A \leftrightarrow \neg D$ 2, 3 $\leftrightarrow$E
5. $A$ 1 &E
6. $\neg D$ 4, 5 $\leftrightarrow$E
7. $\neg D \lor E$ 6 $\lor$I
Instructions: Complete the following derivations by entering the appropriate justifications.
(1a) Derive: \((A \rightarrow B) \& (A \rightarrow \neg D)\)

1. \(A \rightarrow (B \& \neg D)\) Assumption

2. \(A\) \(\rightarrow\)I

3. \(B \& \neg D\) 1, 2 \(\rightarrow\)E

4. \(B\) 3 \&E

5. \(A \rightarrow B\) 2-4 \(\rightarrow\)I

6. \(A\) \(\rightarrow\)I

7. \(B \& \neg D\) 1, 6 \(\rightarrow\)E

8. \(\neg D\) 7 \&E

9. \(A \rightarrow \neg D\) 6-8 \(\rightarrow\)I

10. \((A \rightarrow B) \& (A \rightarrow \neg D)\) 5, 9 \&I
(1b) Derive: $A \rightarrow [B \rightarrow (C \lor D)]$

1. $(A \& B) \rightarrow C$ Assumption
2. $A$ $A / \rightarrow I$
3. $B$ $A / \rightarrow I$
4. $A \& B$ 2, 3 &I
5. $C$ 1, 4 →E
6. $C \lor D$ 5 VI
7. $B \rightarrow (C \lor D)$ 3-6 →I
8. $A \rightarrow [B \rightarrow (C \lor D)]$ 2-7 →I
Instructions: Complete the following derivations.
(2a) Derive: A ↔ B

1. A  
   Assumption
2. B  
   Assumption
3. \[ \text{A} \]  
   A / ↔I
4. \[ \text{B} \]  
   2 R
5. \[ \text{B} \]  
   A / ↔I
6. \[ \text{A} \]  
   1 R
7. A ↔ B  
   3-4, 5-6 ↔I
(2b) Derive: \( \neg B \)

1. \( B \rightarrow \neg B \)  
   Assumption

2. \( B \)  
   A / \( \neg I \)

3. \( \neg B \)  
   1, 2 \( \rightarrow E \)

4. \( B \)  
   2 \( R \)

5. \( \neg B \)  
   2-4 \( \neg I \)
(2c) Derive: A

1. \( \neg\neg A \) Assumption
2. \( \neg A \) A / \( \neg E \)
3. \( \neg\neg A \) 1 R
4. \( \neg A \) 2 R
5. A 2-4 \( \neg E \)
Derivation Rules of Thumb
Reasoning backwards

• If the ultimate goal sentence is atomic, the final step in the derivation will see the application of an elimination rule.

**NB:** In special cases like establishing inconsistency that final rule may simply be reiteration.

• If it’s a compound sentence, the final step in the derivation may involve either an introduction or an elimination rule.

**NB:** There are generally multiple ways to derive the same conclusion from the same premises.

• We can apply the same reasoning to the 2\textsuperscript{nd}-to-last, 3\textsuperscript{rd}-to-last, ... sentence until we close the premises-conclusion gap.
To subderive or not to subderive

• Another rule of thumb is to ask whether the ultimate goal sentence is a constituent part of a complex sentence premise.

• Such parts can take the form of conjuncts, disjuncts, consequents and either side of a biconditional.

• If so, the need for a subderivation for that sentence is unlikely. What is needed instead is an elimination rule.

**NB**: Disjunction elimination is the exception here as it’s both an elimination rule and employs subderivations.

• Otherwise, it is advisable to consider a subderivation.
The End