# Formal Logic

## Lecture 8: The Semantics of Predicate Logic (Part I)

Dr. Ioannis Votsis

ioannis.votsis@nchlondon.ac.uk

www.votsis.org

# Some Ideas from Set Theory

# Sets

- A set is simply a collection of things. We call the latter their *elements* or *members*.

- If element $\alpha$ belongs to set $A$, we can express this as: $\alpha \in A$. If element $\beta$ does not belong to $A$, then we express it as: $\beta \notin A$

- We express the contents of each set within curly brackets.

  $A$ = {Dee, Joe, Alex, Kate}
  $B$ = {Ann, David, Mary, Tim}

- The order in which the elements appear doesn't matter in normal sets. We could have written $A$ = {Kate, Alex, Dee, Joe}.

# Ordered sets

- An ordered set, a.k.a. an '*n*-tuple', is a set whose members are identified by pointy brackets < >.

- Each member of an *n*-tuple contains exactly *n* objects that are ordered from left to right and separated by commas.

**Examples of individual members of ordered sets**:

<a, b> ≠ <b, a>

<1, 2> ≠ <2, 1>

<a, b, c> ≠ <b, c, a> ≠ <a, c, b> ≠ ...

**Examples of ordered sets**:

$S_1$ = {<a, b>}

$S_2$ = {<a, b>, <b, a>}

$S_3$ = {<1, 2, 3>, <3, 2, 1>}

# Ordered sets and relations

- It should be clear that *n*-tuples come in pairs (*n*=2), triples (*n*=3), quadruples (*n*=4), quintuples (*n*=5), and so on.

- An *n*-tuple can be used to capture all the things to which *n*-place relations apply.

A binary, i.e. 2-place, relation contains only pairs.
**Example**: ‘x is smaller than y’ $S_1$ = {<0, 1>, <1, 2>, <2, 3>, ... }

A ternary, i.e. 3-place, relation contains only triples.
**Example**: ‘x is between y and z’ $S_1$ = {<1, 0, 2>, <2, 1, 3>, ... }

and so on for more complex relations.

**NB**: 1-place relations are captured by ‘unordered’ sets.

# Ordered sets: Example 1

- Actually, to specify an ordered set, we first need to specify a set from which the objects of the ordered set are drawn.

- We call this the universe (a.k.a. domain) of discourse (UD).

UD: {Ann, Alex, Dee, David, Tim, Mary, Joe, Kate}
T: {<Tim, Alex>, <Alex, Kate>, <Kate, Joe>, <Joe, Mary>, <Mary, Ann>, <Ann, David>, <David, Dee>}

- T may in fact be capturing the relation 'x is taller than y'.

**NB**: T contains pairs. As specified above, it relates all the objects in UD but that's not necessary, e.g. T': {<Tim, Ann>}.

# Ordered sets: Example 2

- Suppose our UD consists of some boxers and we're interested in the relations 'x is a heavyweight' and 'x beat y'.

  UD: {Fury, Wilder, Joshua, Klitschko, Mayweather, Pacquiao}

  H: {Fury, Wilder, Joshua, Klitschko}
  B: {<Fury, Wilder>, <Joshua, Klitschko>, <Mayweather, Pacquiao>}

  where:
  H captures the 1-place relation 'x is a heavyweight'.
  B captures the 2-place relation 'x beat y'.

# Ordered sets: Example 3

- Suppose our UD consists of a bunch of different objects and we're interested in the relation 'x is identical to y'.

  UD: {Ioannis, Brian, David, Christoph, Naomi, 1, 2, 3, 4, 5}

  I: {<Ioannis, Ioannis>, <Brian, Brian>, <David, David>, <Christoph, Christoph> <Naomi, Naomi>, <1, 1>, <2, 2>, <3, 3>, <4 , 4>, <5, 5>}

  where:
  I captures the 2-place relation 'x is identical to y'.

  **NB**: One of the properties of relation I is reflexivity. That's because all objects in UD bear that relation to themselves.

# The Semantics of Predicate Logic

# Interpretations

- Recall that in propositional logic, we gave 'meaning' to whole sentences by assigning truth-values to them.
- In predicate logic, there is a similar semantic notion at play, though it is a little more complex.
- An interpretation does not rely on natural language articulations of predicates.
- Instead, set theoretical structures, or more simply *structures*, are employed to give 'meaning'.

# Extensions

- A structure gives ‘meaning’ by providing extensions to the predicates and constants of $L_2$.

- What is an extension? It is a set of objects which satisfies the given predicate or constant.

- Here’s the extension of the predicate S which in English is expressed by ‘... is a student of NCH Logic Class 2019-2020’:

  S: {Cormac, David, Eirin, Isaac, Jeremy, Max, Mohamed, Phi}

- Here’s the extension of the constant i: {Ioannis}. Constants always have one object as their extension.

# Interpretations as structures

- An interpretation in $L_2$ is provided by an $L_2$-structure that assigns extensions to UD, predicates and constants as follows:

  (1) The universe of discourse UD is assigned at least one object, i.e. it must not be empty.

  (2) Each *n*-place predicate is assigned an *n*-tuple from (i.e. which ranges over) the elements of UD.

  (3) Each constant is assigned one element from UD.

  **NB**: An interpretation also assigns truth-values to sentences (i.e. 0-place predicates) of $L_2$.

# Interpretations: Some qualifications

- An interpretation may assign the same member of UD to one or more constants, e.g. a: 1, c: 1.

  **NB**: This does justice to the fact that in natural language we use different names to denote the same individual.

- When the UD is restricted to a certain class, e.g. natural numbers, we can read the quantifiers accordingly.

  UD: The set of natural numbers
  P: The set of prime numbers,  S: {<x, y>: x is smaller than y}

  $(\forall x)(Px \rightarrow (\exists y) Syx)$
  Every natural number x is such that if x is a prime then there is at least one natural number y smaller than x.

# Fixing truth-values: Example 1

UD = {1, 2, 3, 4}

E: {2, 4}

G: {<2, 1>, <3, 1>, <4, 1>, <3, 2>, <4, 2>, <4, 3>, <1, 1>, <2, 2>, <3, 3>, <4, 4>}

P: {<2, 3, 4>, <1, 2, 4>}

a: 1, b: 2, c: 3, d: 4

| **True Sentences** | **Because** |
| --- | --- |
| Eb | 2 ∈ E |
| ¬Ea | It is false that 1 ∈ E |
| Gdc | <4, 3> ∈ G |
| Pbcd | <2, 3, 4> ∈ P |

**NB**: As it so happens, G stands for ‘x is greater than or equal to y’ and P stands for ‘the sum of x, y, and z is odd’.

# Fixing truth-values: Example 2

UD = {1, 2, 3, 4}
E: {2, 4}
G: {<2, 1>, <3, 1>, <4, 1>, <3, 2>, <4, 2>, <4, 3>, <1, 1>, <2, 2>, <3, 3>, <4, 4>}
P: {<2, 3, 4>, <1, 2, 4>}
a: 1, b: 2, c: 3, d: 4

| **False Sentences** | **Because** |
| --- | --- |
| Ea | $1 \notin E$ |
| ¬Ed | It is true that $4 \in E$ |
| Gac | $\langle 1, 3 \rangle \notin G$ |
| Pabc | $\langle 1, 2, 3 \rangle \notin P$ |

**NB**: As it so happens, G stands for ‘x is greater than or equal to y’ and P stands for ‘the sum of x, y, and z is odd’.

# Fixing truth-values: Example 3

UD: {Ann, Tim, Mary, Joe, Kate}

T: {<Tim, Kate>, <Kate, Joe>, <Joe, Mary>, <Mary, Ann>}

a: Ann, j: Joe, k: Kate, m: Mary, t: Tim

**True or False?**

Taj

¬Tmt

¬Tjm

Ttk

Taj & Ttk

¬Tmt V ¬Tjm

**NB**: As it so happens, T stands for ‘x is taller than y’.

# Fixing truth-values: Example 3 (continued)

UD: {Ann, Tim, Mary, Joe, Kate}
T: {<Tim, Kate>, <Kate, Joe>, <Joe, Mary>, <Mary, Ann>}
a: Ann, j: Joe, k: Kate, m: Mary, t: Tim

| **True or False?** | **Result** | **Because** |
|---|---|---|
| Taj | False | <Ann, Joe> ∉ T |
| ¬Tmt | True | It is false that <Mary, Tim> ∈ T |
| ¬Tjm | False | It is true that <Joe, Mary> ∈ T |
| Ttk | True | <Tim, Kate> ∈ T |
| Taj & Ttk | False | <Ann, Joe> ∉ T |
| ¬Tmt V ¬Tjm | True | It is false that <Mary, Tim> ∈ T |

**NB**: As it so happens, T stands for ‘x is taller than y’.

# Fixing truth-values: Example 4

UD: {Tom, Dick, Harry}

B: {<Tom, Dick>, <Tom, Harry>, <Tom, Tom>, <Dick, Harry>}

R: {Dick, Harry}

P: {Harry}

d: Dick, h: Harry, t: Tom

**True or False?**

$(\forall x)\ Rx$

$(\forall z)\ (Pz \rightarrow Rz)$

$(\forall x)(\forall y)\ \neg Bxy$

$(\forall x)(\forall y)\ (Bxy \rightarrow Ry) \& Ph$

**NB**: As it so happens, B stands for ‘x is the boss of y’, R stands for ‘x gets a raise’ and P stands for ‘x parties’.

# Fixing truth-values: Example 4 (continued)

UD: {Tom, Dick, Harry}
B: {<Tom, Dick>, <Tom, Harry>, <Tom, Tom>, <Dick, Harry>}
R: {Dick, Harry}
P: {Harry}
d: Dick, h: Harry, t: Tom

| **True or False?** | **Result** | **Because** |
|---|---|---|
| $(\forall x)\ Rx$ | False | Tom $\notin$ R |
| $(\forall z)\ (Pz \rightarrow Rz)$ | True | Harry $\in$ P and Harry $\in$ R |
| $(\forall x)(\forall y)\ \neg Bxy$ | False | e.g. <Tom, Dick> $\in$ B |
| $(\forall x)(\forall y)\ (Bxy \rightarrow Ry) \& Ph$ | False | <Tom, Tom> $\in$ B and Tom $\notin$ R |

**NB**: As it so happens, B stands for ‘x is the boss of y’, R stands for ‘x gets a raise’ and P stands for ‘x parties’.

# Fixing truth-values: Example 5

UD: {Tom, Dick, Harry}

B: {<Tom, Dick>, <Tom, Harry>, <Tom, Tom>, <Dick, Harry>}

R: {Dick, Harry}

P: {Harry}

d: Dick, h: Harry, t: Tom

**True or False?**

$(\exists z)\ Pz\ \&\ Rd$

$(\exists y)(\exists z)\ Byz$

$\neg(\exists y)\ (Ry\ \&\ Py)$

$(\exists x)(\forall y)\ Bxy \vee Rt$

**NB**: As it so happens, B stands for ‘x is the boss of y’, R stands for ‘x gets a raise’ and P stands for ‘x parties’.

# Fixing truth-values: Example 5 (continued)

UD: {Tom, Dick, Harry}

B: {<Tom, Dick>, <Tom, Harry>, <Tom, Tom>, <Dick, Harry>}

R: {Dick, Harry}

P: {Harry}

d: Dick, h: Harry, t: Tom

| True or False? | Result | Because |
|---|---|---|
| (∃z) Pz & Rd | True | Harry ∈ P and Dick ∈ R |
| (∃y)(∃z) Byz | True | e.g. <Tom, Dick> ∈ B |
| ¬(∃y) (Ry & Py) | False | It is true: Harry ∈ R and Harry ∈ P |
| (∃x)(∀y) Bxy ∨ Rt | True | <Tom, Dick> ∈ B, <Tom, Harry> ∈ B and <Tom, Tom> ∈ B |

The End