Formal Logic

Lecture 9: The Semantics of Predicate Logic (Part II)

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Validity and other Notions in L_2

Logical truth in L_2

- A sentence ϕ in L_2 is *logically true* IFF it is true under any interpretation.

  NB: Bergmann et al. (2014) call it 'quantificationally true'.

  Examples:
  $(\forall x)(Px \rightarrow Px)$
  $(\exists x)(Px \vee \neg Px)$

- We cannot show that a sentence is logically true via one or even some interpretations. We need all of them.

- We can show that a sentence is *not* logically true by coming up with one interpretation that makes the sentence false.

Contradiction in L_2

- A sentence ϕ in L_2 is *logically false* IFF it is false under any interpretation.

  NB: Bergmann et al. (2014) call it 'quantificationally false'.

  Examples:
  $\neg(\exists x)(Px \vee \neg Px)$
  $(\forall x)\ Px\ \&\ \neg Pa$

- We cannot show that a sentence is logically false via one or even some interpretations. We need all of them.

- We can show that a sentence is *not* logically false by coming up with one interpretation that makes the sentence true.

Contingency in L_2

- A sentence ϕ in L_2 is *contingent* if and only if it is neither logically true nor logically false.

  NB: Bergmann et al. (2014) call it 'quant. indeterminate'.

  Examples:
  $(\exists x)(Px \,\&\, Qx)$
  $(\forall y)\ Ray$

- We can show that a sentence is contingent via two interpretations (one where it comes out true and one false).

Validity in L_2

- An argument in L_2 is *valid* IFF there is no interpretation under which the premises are true and the conclusion is false.

  NB: Bergmann et al. (2014) call it 'quantificationally valid'.

  Examples:

$(\forall x)\ Px$	entails	Pa
$\{(\forall x)(Px \rightarrow Qa), (\exists x)Px\}$	entails	Qa

- We <u>cannot show</u> that an argument is valid via one or even some interpretations. We need all of them.

- We <u>can show</u> that an argument is invalid via 1 interpretation that makes the premises true and the conclusion false.

Consistency in L_2

- A set of sentences in L_2 is *consistent* IFF there is at least one interpretation on which all the members of the set are true.

  NB: Bergmann et al. (2014) call them ‘quant. consistent’.

  Examples:
  {(∃x)Hx, ¬Hc}
  {(∀y)Ray, ¬Rba ∨ (∃z) ¬Raz}

- We cannot show that a set of sentences is inconsistent via one or even some interpretations. We need all of them.

- We can show that a set of sentences is consistent via one interpretation that makes all sentences true.

Logical equivalence in L_2

- Two sentences ϕ, ψ in L_2 are *logically equivalent* IFF there is no interpretation on which ϕ, ψ have different truth-values.

  NB: Bergmann et al. (2014) call them 'quant. equivalent'.

  Examples:

$(\forall x)\ \neg Qx$	and	$\neg(\exists x)\ Qx$
$(\exists x)(Ax\ \&\ \neg Bx)$	and	$\neg(\forall x)(Ax \rightarrow Bx)$

- We <u>cannot show</u> that a pair of sentences is equivalent via one or even some interpretations. We need all of them.

- We <u>can show</u> that a pair is *not* logically equivalent via 1 interpretation where one sentence is true and the other false.

The undecidability of L_2

- Alonzo Church (1936) proved a fundamental limit to first-order predicate logic.

- There is no decision procedure that determines for every group of L_2 sentences whether they are:

  * logically true, logically false or contingent
  * consistent or inconsistent
  * valid or invalid
  * equivalent or inequivalent

  NB: In L_1, there is such a procedure, namely truth-tables.

- **Decision procedure**: A mechanical method that shows for all cases & in a finite number of steps whether a property holds.

Exercise Set #5

Exercise 4.1

- **Instructions**: Determine whether the following expressions are formulae of L_2 and say which of those are also sentences of L_2.

Exercise 4.1

Formula, sentence or neither?

(i) $(\forall x)\ (P_1 x \rightarrow Qy)$

(ii) $(\exists x)\ \neg(\neg\neg(\exists y)Py\ \&\ \neg\neg\neg\neg\neg Rxa)$

(iii) P

(iv) $(\forall x)(\exists y)(\exists z)\ (R_{25}xyz)$

(v) $(\forall x)(\exists x)\ Qxx$

(vi) $\neg(\neg((\exists x)Px\ \&\ (\exists y)Qy))$

(vii) $(\forall x)(\exists y(Pxy\ \&\ Px) \vee Qxyx)$

Exercise 4.1: Solution

Formula, sentence or neither?

(i) $(\forall x)\ (P_1x \to Qy)$	**Formula (y is free)**
(ii) $(\exists x)\ \neg(\neg\neg(\exists y)Py\ \&\ \neg\neg\neg\neg\neg Rxa)$	**Sentence**
(iii) P	**Sentence**
(iv) $(\forall x)(\exists y)(\exists z)\ (R_{25}xyz)$	**Neither (brackets on R)**
(v) $(\forall x)(\exists x)\ Qxx$	**Sentence**
(vi) $\neg(\neg((\exists x)Px\ \&\ (\exists y)Qy))$	**Neither (outer brackets)**
(vii) $(\forall x)((\exists y)(Pxy\ \&\ Px) \vee Qxyx)$	**Formula (2^{nd} y is free)**

Exercise 4.3

- **Instructions**: Provide L_2-formalisations for the following English sentences.

Exercise 4.3

English Sentences:

(i) London is big and ugly.
(ii) Culham is a large village.
(iii) A city has a city hall.
(iv) Material objects are divisible.
(v) Tom owns at least one car.
(vi) Tom owns at least one car and he won't sell it.
(vii) One man has visited every country.

Exercise 4.3: Solution

English Sentences:	**Formalised:**
(i) London is big and ugly.	Ba&Ua
(ii) Culham is a large village.	Lc
(iii) A city has a city hall.	$(\forall x)(Cx \rightarrow (\exists y)(Pxy \& Hy))$
(iv) Material objects are divisible.	$(\forall x)(Mx \rightarrow Dx)$
(v) Tom owns at least one car.	$(\exists x)(Otx \& Cx)$
(vi) Tom owns at least one car and he won't sell it.	$(\exists x)((Otx \& Cx) \& \neg Stx)$
(vii) One man has visited every country.	$(\exists x)(Mx \& (\forall y)(Cy \rightarrow Vxy))$

NB: Other readings sometimes possible, e.g. one means one in vii.

Exercise 4.4

- **Instructions**: Translate the L_2-sentences below into English using the following dictionary.

Exercise 4.4

- Translate the L_2-sentences below into English using the following dictionary.

a: Tom
P^1: ... is a person
Q^1: ... acts freely

L_2 Sentences:

(i) Qa
(ii) $(Qa \vee \neg Pa)$
(iii) $(\forall x)(Px \rightarrow Qx)$
(iv) $(\forall x)(Px \leftrightarrow Qx)$
(v) $\neg \exists z_1 Q z_1$

Exercise 4.4: Solution

- Translate the L_2-sentences below into English using the following dictionary.

  a: Tom
  P^1: ... is a person
  Q^1: ... acts freely

L_2 **Sentences:**	**English Sentences:**
(i) Qa	Tom acts freely.
(ii) $(Qa \vee \neg Pa)$	Tom acts freely or Tom is not a person.
(iii) $(\forall x)\ (Px \rightarrow Qx)$	Every person acts freely.
(iv) $(\forall x)\ (Px \leftrightarrow Qx)$	Every person acts freely and vice-versa.
(v) $\neg \exists z_1 Q z_1$	Nothing acts freely.

Exercise 5.1

- **Instructions**: Consider an L_2-structure S... Are the following sentences true or false in this structure? Sketch proofs...

Exercise 5.1

- Consider an L_2-structure S... Are the following sentences true or false in this structure? Sketch proofs...

  UD: {1, 2, 3}, P: {2}, R: {<1, 2>, <2, 3>, <1, 3>}, a: 1, b: 3

  (i) Pa
  (ii) Rab
  (iii) Rba
  (iv) Rab ↔ Rba
  (v) Rbb ∨ (¬Pa & ¬Raa)
  (vi) (∃x)Rax
  (vii) (∃x)(Rax & Rxb)
  (viii) Pb ∨ (∃x)Rxx
  (ix) (∀x)(∃y)Rxy

Exercise 5.1: Solution

- Consider an L_2-structure S… Are the following sentences true or false in this structure? Sketch proofs…

  UD: {1, 2, 3}, P: {2}, R: {$<1, 2>, <2, 3>, <1, 3>$}, a: 1, b: 3

(i) Pa	False as $1 \notin P$
(ii) Rab	True as $<1, 3> \in R$
(iii) Rba	False as $<3, 1> \notin R$
(iv) Rab $\leftrightarrow$ Rba	False as Rab is true and Rba false
(v) Rbb $\vee$ ($\neg$Pa & $\neg$Raa)	True as $1 \notin P$ and $<1, 1> \notin R$
(vi) $(\exists x)$Rax	True, e.g. $<1, 2> \in R$
(vii) $(\exists x)$(Rax & Rxb)	True as $<1, 2> \in R$ and $<2, 3> \in R$
(viii) Pb $\vee$ $(\exists x)$Rxx	False as neither $3 \in P$ nor $<x, x> \in R$
(ix) $(\forall x)(\exists y)$Rxy	False as $<3, y> \notin R$

The End