Philosophy of Science

Lecture 4: Probability and Bayesianism
Special Topic: Novel Predictions

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Set Theory: A Refresher
A set is simply a collection of things. We call the latter their *elements* or *members*.

If element $\alpha$ belongs to set $A$, we can express this as: $\alpha \in A$

We express the contents of each set within curly brackets.

$A = \{1, 3, 5, 7\}$
$B = \{2, 4, 6, 8\}$

The order in which the elements appears doesn’t matter in regular sets. We could have written $A = \{7, 5, 3, 1\}$.

**NB:** In so-called ‘ordered sets’, by contrast, this does matter!
Operators

• In the same way that arithmetic has binary operators -, +, × and ÷, set theory has its own binary operators, e.g. \( \cap \) and \( \cup \).

\( A \cap B \) expresses the *intersection* (or overlap) between A and B.

\( A \cup B \) expresses the *union* between A and B.

Suppose that, as before, \( A = \{1, 3, 5, 7\} \) and \( B = \{2, 4, 6, 8\} \).

Then:

\( A \cap B = \emptyset \) where \( \emptyset \) denotes the empty-set.

\( A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\} \)
Venn diagrams

- We can visually express these relations via Venn diagrams.

Mutually Exclusive

- $D$ is the set of all things (‘the universe’).

- Compare the two on the left: top: $A \cup B \neq D$
  bottom: $A \cup \neg A = D$
Probability Theory
Probability theory

• A branch of mathematics, it studies the likelihood of events.

• Deals with indeterministic processes and/or incomplete info.

• The study begins in the 17th century with games of chance.

• Correspondence between Blaise Pascal and Pierre de Fermat.

• First axiomatised by Andrey Kolmogorov in 1933.

• Referred to as the ‘probability calculus’.
The axioms (unconditional probability)

- $A, B, \ldots$ are propositions expressing possible outcomes or events, hypotheses, etc. (in relation to an experiment).

$S$, ‘the sample space’, is the set of all propositions (outcomes).

- The axioms:

1. $0 \leq P(A) \leq 1$ for all $A \in S$

2. If $A$ is a logical/necessary truth, $P(A) = 1$
   **NB**: If $A$ is a contradiction/impossibility, $P(A) = 0$.

3. If $A$ and $B$ are mutually exclusive, $P(A \lor B) = P(A) + P(B)$.
   **NB1**: We can also write this as $P(A \cup B) = P(A) + P(B)$.
   **NB2**: This is known as the *special addition rule*. 
Toy example: Special addition rule

• Randomly drawing 1 card (no jokers).

\[ P(\text{drawing spades } \spadesuit) = \frac{13}{52} = \frac{1}{4} = 0.25 \]

\[ P(\text{drawing spades } \spadesuit \text{ or hearts } \heartsuit) = \frac{13}{52} + \frac{13}{52} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = 0.5 \]

• Randomly rolling a die once (die cannot land on its corners).

\[ P(\text{four}) = \frac{1}{6} = 0.1666... \]

\[ P(\text{four or two}) = \frac{1}{6} + \frac{1}{6} = 0.3333... \]
Independence

- The idea here is that the one event occurring doesn’t affect the probability of the other event occurring.

- Independence can be expressed as follows: \( P(B|A) = P(B) \). That means dependence is expressed as \( P(B|A) \neq P(B) \).

Examples:

\[
P(\text{Lung Cancer} | \text{Smoke}) > P(\text{Lung Cancer}) \quad \text{dependent}
\]

\[
P(\text{Lung Cancer} | \text{Brexit}) = P(\text{Lung Cancer}) \quad \text{independent}
\]
Special multiplication rule

• To ask what’s the probability of two (or more) events occurring in a row is to ask \( P(A & B) = P(A \cap B) = ? \)

• Assuming independence, we apply the special multiplication rule \( P(A \cap B) = P(A) \times P(B) \) to answer this question.

• Randomly drawing two cards in a row (w/replacement).

\[
P(\text{drawing spades } \spadesuit \text{ and hearts } \heartsuit) = ?
\]
\[
\frac{13}{52} \times \frac{13}{52} = \frac{1}{4} \times \frac{1}{4} = 0.0625
\]

• Randomly rolling a die twice in a row.

\[
P(\text{four and two}) = ?
\]
\[
\frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0277...
\]
• If there’s a special rule, then there’s a general rule. The general multiplication rule holds: \( P(A \cap B) = P(A) \times P(B|A) \).

• Recall that the special multiplication rule requires independence, i.e. that \( P(B|A) = P(B) \).

• That means that the last term in the general multiplication rule, i.e. \( P(B|A) \), is replaceable by \( P(B) \).

General Multiplication Rule: \( P(A \cap B) = P(A) \times P(B|A) \)

Special Multiplication Rule: \( P(A \cap B) = P(A) \times P(B) \)
Useful theorems: Negation and equivalence

**Negation:** $P(\neg A) = 1 - P(A)$. Equivalently: $P(\neg A) + P(A) = 1$.

**Equivalence:** If $A$ and $B$ are logically equivalent, $P(A) = P(B)$.
Useful theorems: Implication

**Implication:** If $A$ logically entails $B$, then $P(B) \geq P(A)$.

**NB:** In this diagram $P(B) > P(A)$.

**NB:** In this diagram $P(B) = P(A)$. 
Useful theorems: General and special addition

**General addition:** $P(A \lor B) = P(A) + P(B) - P(A \& B)$

**NB:** In this diagram $P(A \& B) \neq 0$, i.e. $A, B$ are not mutually exclusive.

We subtract $P(A \& B)$ as we only want to count that area once.

**Special addition:** $P(A \lor B) = P(A) + P(B)$.

**NB:** In this diagram $P(A \& B) = 0$, i.e. $A, B$ are mutually exclusive.

The last term of general addition rule can thus be omitted.
Conditional probabilities

• $P(A|B)$ is read as ‘the probability of $A$, given $B$ (is true)’.

• **Definition:**
  
  $$P(A|B) = \frac{P(A \& B)}{P(B)}$$

  where $P(B) > 0$.

**NB:** In some axiomatisations such probabilities are primitive.

• The importance of conditional probabilities becomes clear when it is recognised that we can ask questions like:

  * $P(T_1|E_1) = ?$
  * $P(E_1|T_1) = ?$
Bayesian Confirmation Theory
• From the probability axioms one can derive Bayes Theorem:

\[ P(H|E) = \frac{P(E|H) \times P(H)}{P(E)} \]

where \( P(E) > 0 \).

\( P(H|E) \) is the posterior probability of the hypothesis.  
\( P(E|H) \) is the likelihood of the evidence.  
\( P(H) \) is the prior probability of the hypothesis.  
\( P(E) \) is the prior probability of the evidence.
Bayesianism, conditionalisation and relevance

- Bayes theorem on its own is not an account of confirmation.

- **Bayesianism** is such an account. It takes Bayes theorem as a starting point and adds the following epistemic principle:

  **The (simple) principle of conditionalisation** (a.k.a. the ‘updating rule’): \( P_{\text{new}}(H) = P_{\text{initial}}(H \mid E) = p \)

- That means the new prior probability of \( H \) is updated to be the old posterior probability of \( H \) given \( E \).

- **The relevance criterion of confirmation:**
  * \( E \) confirms \( H \) if and only if \( P(H \mid E) > P(H) \)
  * \( E \) disconfirms \( H \) if and only if \( P(H \mid E) < P(H) \)
  * \( E \) is neutral to \( H \) if and only if \( P(H \mid E) = P(H) \)
Confirmation measures

• To turn Bayesianism into a full-blown quantitative account of confirmation we need a **measure**.

• That is, we need a way to quantify the degree of confirmation a piece of evidence lends to a hypothesis.

• Several have been proposed (see Crupi et al 2007):

<table>
<thead>
<tr>
<th>TABLE 1. ALTERNATIVE BAYESIAN MEASURES OF CONFIRMATION.</th>
</tr>
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<tbody>
<tr>
<td>$D(e, h) = p(h</td>
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<tr>
<td>$S(e, h) = p(h</td>
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<tr>
<td>$M(e, h) = p(e</td>
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<tr>
<td>$N(e, h) = p(e</td>
</tr>
<tr>
<td>$C(e, h) = p(e &amp; h) - p(e) \cdot p(h)$</td>
</tr>
<tr>
<td>$R(e, h) = \frac{p(h</td>
</tr>
<tr>
<td>$G(e, h) = 1 - \frac{p(\neg h</td>
</tr>
<tr>
<td>$L(e, h) = \frac{p(e</td>
</tr>
</tbody>
</table>

Carnap ([1950] 1962)
Christensen (1999)
Mortimer (1988)
Nozick (1981)
Carnap ([1950] 1962)
Finch (1960)
Rips (2001)
Kemeny and Oppenheim (1952)
Bayesianism has numerous applications. For example, there are Bayesianist branches of the following theories:

**Confirmation theory** – Seeks to analyse relations of support between evidence and theory.

**Formal learning theory**: Mathematical treatment of how an agent learns and in particular how they ought to learn.

**Decision theory** – Mathematical treatment of (rational) behaviour and choice-making in the presence of options.
Eliciting degrees of belief

- Suppose probabilities express an agent’s (rational) degrees of belief in propositions, events, etc.

- How do we elicit such degrees of belief? One suggestion is through a *willingness to bet*!

  If I have a high confidence that $A$ is true, then I would presumably be very willing to bet on it.

  My degree of belief should be high: $P(A) \approx 1$.

- Some, Jeffrey (1983), object to this betting approach:

  Irrational to bet that ‘everyone dies next year’. If I win I can’t collect. So cannot always elicit degrees of belief via betting.
How to choose priors: Two schools of thought

• How do we assign values to $P(H)$ and $P(E)$? Probability theory provides few constraints, e.g. values from 0 to 1.

• There are two schools of thought within Bayesianism.

  (1) **Subjective Bayesians**: Over and above those constraints, our choice of priors is a subjective matter.

   **Supporters**: De Finetti, Howson, Jeffrey and Ramsey.

  (2) **Objective Bayesians**: Choice of priors can be further constrained, e.g. the principle of indifference.

   **Supporters**: Jaynes, Rosenkrantz and Williamson.
Dutch-book arguments

- **Dutch-book**: A combination of wagers that guarantees a loss through the violation of the axioms of probability.

- Offered not only to justify Bayesianism (+ conditionalisation principle) but also the axioms of probability.

**Example** (synchronic case):
Suppose person $K$ is willing to pay 51p for a wager that returns £1 and 50p for the contrary wage that returns £1.

Taken together the two bets guarantee a 1p loss for $K$ (and a 1p gain for the bookie). In probabilistic terms:

$K$ assigns $P(W)=0.51$ and $P(\neg W)=0.50$. Recall that $P(A \lor \neg A)=1$. $K$ violates this theorem (and axiom) since $P(W \lor \neg W) > 1$. 
Some Positives
Advantage: Confirmation by entailment

- Bayesianism can account for **confirmation by entailment** and thus is meant to do justice to the H-D account.

Suppose: $H \vdash E$. That means: $P(E|H) = 1$

This simplifies our equation to: $P(H|E) = \frac{P(H)}{P(E)}$.

- Recall that if $H \vdash E$ then $P(E) \geq P(H)$. In short, it could not be that $P(E) < P(H)$. This would violate the first axiom.

- Either $P(E) = P(H)$ in which case $P(H|E) = 1$. Or $P(E) > P(H)$ in which case $P(H|E) > P(H)$. In both cases we get confirmation.
Advantage: Unexpected evidence

- It can account for the intuition that unexpected evidence has a higher confirmational value than expected evidence.

- Proposal 1: The more surprising, the lower its probability, i.e. $P(E)$ is close to zero.

- That’s actually not enough to guarantee $P(H|E) > P(H)$. Suppose $P(H)=0.5$, $P(E|H)=0.1$, $P(E)=0.1$. Then $P(H/E)=0.5$.

- Proposal 2: How about requiring that $P(E|H)>P(E)$?

- Here $H$ gets confirmed but again no guarantee $P(H|E) > P(H)$! Suppose $P(H)=0.5$, $P(E|H)=0.11$, $P(E)=0.1$. Then $P(H/E)=0.55$. 
Advantage: Unexpected evidence (2)

• What we need is $P(E|H) \gg P(E)$!

• This accords well with Popper’s demand that theories entail statements that are risky, i.e. risky predictions.

Suppose $H \vdash E_1, E_2$; $P(E_1|H)=1$; $P(E_2|H)=1$; $P(E_1)=.9$; $P(E_2)=.51$

\[
P(H|E_1) = \frac{1 \times 0.5}{0.9} = 0.5555...\]
\[
P(H|E_2) = \frac{1 \times 0.5}{0.51} = 0.9803...\]
• Bayesianism can provide a solution to the raven paradox.

• $H$: All ravens are black, $E_1$: $a$ is black raven, $E_2$: $b$ is a white sock.

It seems reasonable to assert, especially if $a$ is amongst the first black ravens that we observe, that $P(E_1|H) \gg P(E_1)$.

This is not the case with non-black non-ravens since almost everything is such a thing. So, $P(E_2|H) \approx P(E_2)$.

So, $P(E_1|H)/P(E_1) \gg P(E_2|H)/P(E_2)$. Hence $E_2$ confirms $H$ but much less than $E_1$.

**Rationale:** If the world has finite # of things, the more we eliminate as non-black non-ravens, the more confidence in $H$. 
Some Negatives
The choice of priors

• *Subjective Bayesians*: We are free to choose degrees of belief in a given proposition so long as they are coherent.

• **Problem**: People can assign wildly divergent priors to one and the same hypothesis. How can we confirm anything?

• **Replies**:

  (1) Avoid such extreme and implausible priors.

  (2) Unless they start like that, their posteriors should (eventually) converge as the evidence accumulates.

  **NB**: Called the ‘washing out’ ‘swamping out’ of the priors.
The old evidence problem

• Glymour (1980): Oftentimes scientists appeal to evidence that’s already known to support their theories.

• If evidence is ‘old’, then we might say $P(E) = 1$. That reduces our equation to: $P(H|E) = P(E|H) \times P(H)$.

  If $P(E|H) < 1$, then $P(H|E) < P(H)$ [disconfirmation]
  If $P(E|H) = 1$, then $P(H|E) = P(H)$ [no confirmation].

• Historical example: Mercury’s perihelion was discovered in the 19th c. but confirmed Einstein’s GTR in early 20th c.

• **Reply:** Agent must evaluate $H$ counterfactually (holding all the same beliefs minus $E$). Thus $P(E) < 1$. See Howson (1991).
Special Topic: Novel Predictions
Introduction

• *Novel predictions, novel facts or novel evidence* are in some respect phenomena that are ‘unexpected’ vis-à-vis a theory.

**NB:** This sense is explicated below.

• There is a long history of treating such unexpected phenomena as possessing special confirmational weight.

  * Poisson’s spot
  * Starlight delfection
  * Mercury’s perihelion
  * ...


Accommodationism vs. predictivism

• Two families of views regarding their confirmational value.

**Accommodationism:** Accommodated evidence carries as much confirmational weight as predicted evidence.

**Predictivism:** Accommodated evidence (or even evidence that *could have been accommodated*) carries less/no weight.
Predictivism: Two families

- **Temporal-novelty** (e.g. Maher 1988): Phenomena known after $H$ was formed have more (/the only) evidential weight.

- **Use-novelty** (e.g. Worrall 2002): Phenomena not used in the construction of $H$ have more (/the only) evidential weight.

**Rationale**: A hypothesis could not possibly have been shaped to accommodate phenomena that were:

* unknown prior to its formulation
* not used in its construction

- This rules out that the hypothesis is ad hocly constructed.
The End