Philosophy of Science

Lecture 4: Probability and Bayesianism
Special Topic: Novel Predictions

Dr. Ioannis Votsis

ioannis.votsis@nchlondon.ac.uk
www.votsis.org

Set Theory: A Refresher

Sets

- A set is simply a collection of things. We call the latter their *elements* or *members*.

- If element α belongs to set *A*, we can express this as: $\alpha \in A$

- We express the contents of each set within curly brackets.

  $A = \{1, 3, 5, 7\}$
  $B = \{2, 4, 6, 8\}$

- The order in which the elements appears doesn't matter in regular sets. We could have written A = $\{7, 5, 3, 1\}$.

  NB: In so-called 'ordered sets', by contrast, this does matter!

Operators

- In the same way that arithmetic has binary operators -, +, $\times$ and $\div$, set theory has its own binary operators, e.g. $\cap$ and $\cup$.

  $\boldsymbol{A \cap B}$ expresses the *intersection* (or overlap) between A and B.

  $\boldsymbol{A \cup B}$ expresses the *union* between A and B.

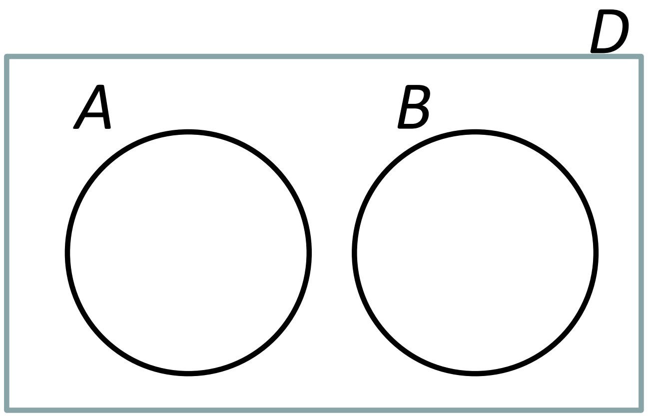
  Suppose that, as before, $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$.

  Then:
  $\boldsymbol{A \cap B} = \emptyset$ where $\emptyset$ denotes the empty-set.

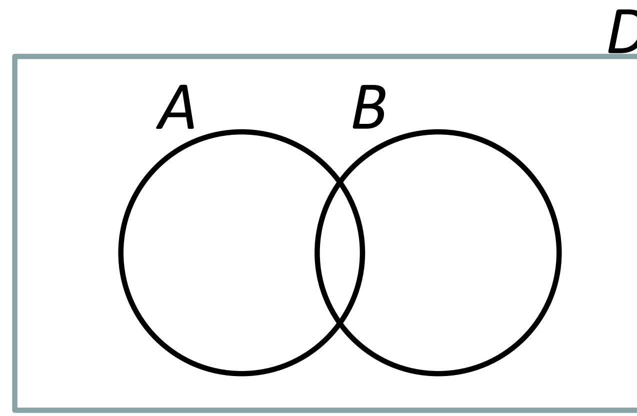
  $\boldsymbol{A \cup B} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Venn diagrams

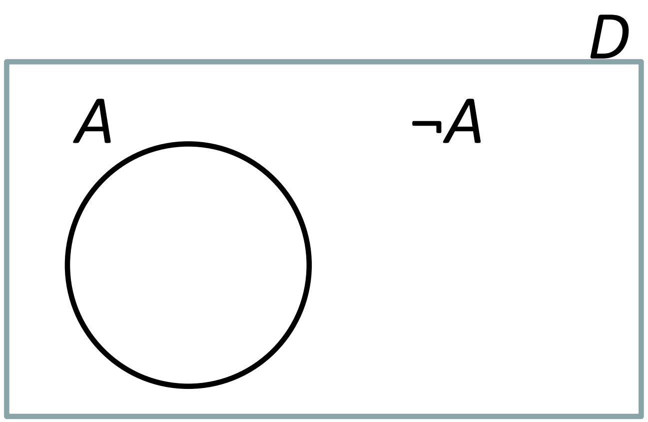
- We can visually express these relations via *Venn diagrams*.



Mutually Exclusive



Non-exclusive



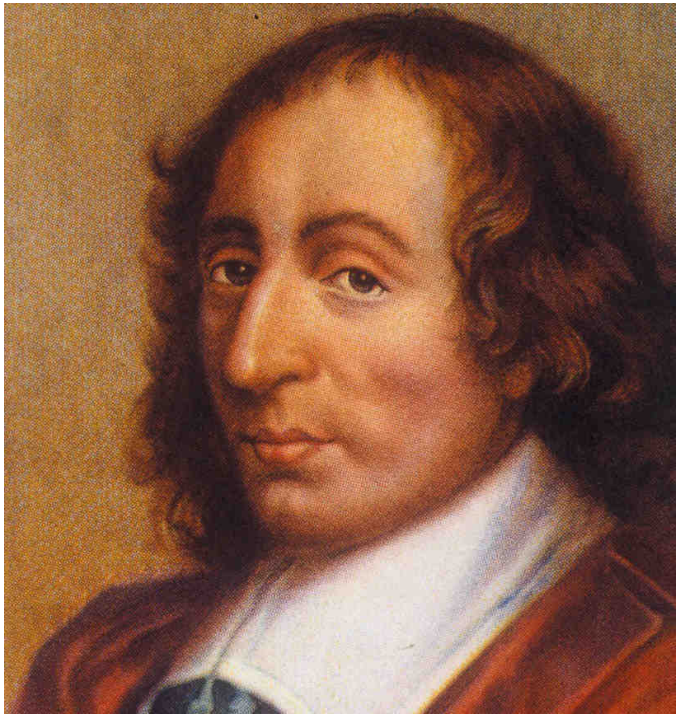
Mutually Exclusive
and Exhaustive

- *D* is the set of all things ('the universe').
- Compare the two on the left:
  top: $A \cup B \neq D$
  bottom: $A \cup \neg A = D$

Probability Theory

Probability theory

- A branch of mathematics, it studies the likelihood of events.
- Deals with indeterministic processes and/or incomplete info.
- The study begins in the 17th century with games of chance.
- Correspondence between Blaise Pascal and Pierre de Fermat.
- First axiomatised by Andrey Kolmogorov in 1933.



- Referred to as the ‘probability calculus’.

The axioms (unconditional probability)

- A, B, ... are propositions expressing possible outcomes or events, hypotheses, etc. (in relation to an experiment).

  S, 'the sample space', is the set of all propositions (outcomes).

- The axioms:

  (1) $0 \leq P(A) \leq 1$ for all $A \in S$

  (2) If A is a logical/necessary truth, $P(A) = 1$
  NB: If A is a contradiction/impossibility, $P(A) = 0$.

  (3) If A and B are mutually exclusive, $P(A \vee B) = P(A) + P(B)$.
  NB1: We can also write this as $P(A \cup B) = P(A) + P(B)$.
  NB2: This is known as the *special addition rule*.

Toy example: Special addition rule

- Randomly drawing 1 card (no jokers).

  P(drawing spades ♠) = ?
  13/52 = ¼ = 0.25

  P(drawing spades ♠ ***or*** hearts ♥) = ?
  13/52 + 13/52 = ¼ + ¼ = ½ = 0.5

- Randomly rolling a die once (die cannot land on its corners).

  P(four) = ?
  1/6 = 0.1666....

  P(four ***or*** two) = ?
  1/6 + 1/6 = 0.3333....

Independence

- The idea here is that the one event occurring doesn't affect the probability of the other event occurring.

- Independence can be expressed as follows: $P(B|A) = P(B)$. That means dependence is expressed as $P(B|A) \neq P(B)$.

  Examples:

  $P(\text{Lung Cancer}|\text{Smoke}) > P(\text{Lung Cancer})$ **dependent**

  $P(\text{Lung Cancer}|\text{Brexit}) = P(\text{Lung Cancer})$ **independent**

Special multiplication rule

- To ask what's the probability of two (or more) events occurring in a row is to ask $P(\text{A} \,\&\, \text{B}) = P(\text{A} \cap \text{B}) = ?$

- Assuming independence, we apply the special multiplication rule $P(\text{A} \cap \text{B}) = P(\text{A}) \times P(\text{B})$ to answer this question.

- Randomly drawing two cards in a row (w/replacement).

  P(drawing spades ♠ ***and*** hearts ♥) = ?
  $13/52 \times 13/52 = \frac{1}{4} \times \frac{1}{4} = 0.0625$

- Randomly rolling a die twice in a row.

  P(four ***and*** two) = ?
  $1/6 \times 1/6 = 1/36 = 0.0277...$

General multiplication rule

- If there's a special rule, then there's a general rule. The general multiplication rule holds: $P(A \cap B) = P(A) \times P(B|A)$.
- Recall that the special multiplication rule requires independence, i.e. that $P(B|A) = P(B)$.
- That means that the last term in the general multiplication rule, i.e. $P(B|A)$, is replaceable by $P(B)$.

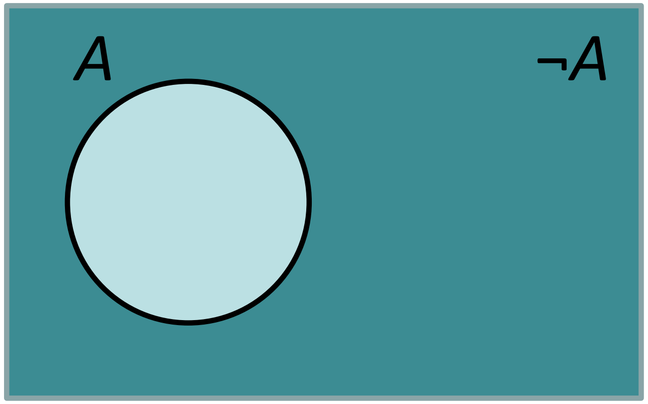
General Multiplication Rule: $P(A \cap B) = P(A) \times \boxed{P(B|A)}$

↓

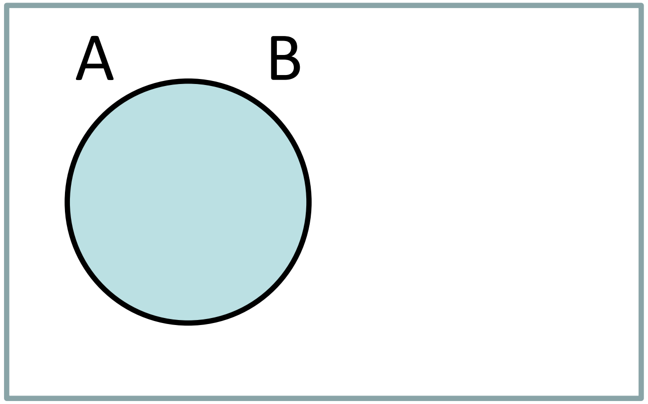
Special Multiplication Rule: $P(A \cap B) = P(A) \times \boxed{P(B)}$

Useful theorems: Negation and equivalence

Negation: $P(\neg A) = 1 - P(A)$. Equivalently: $P(\neg A) + P(A) = 1$.

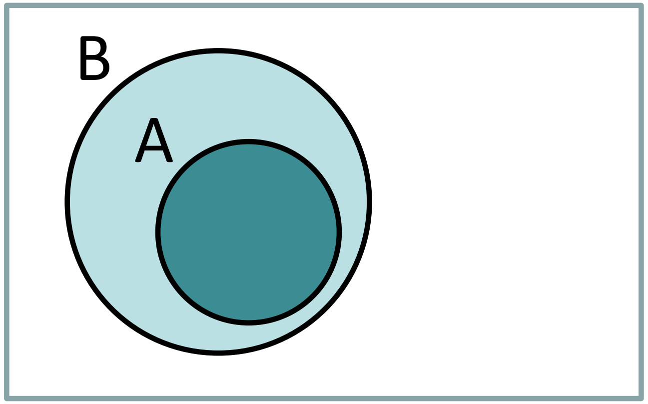


Equivalence: If A and B are logically equivalent, $P(A) = P(B)$.

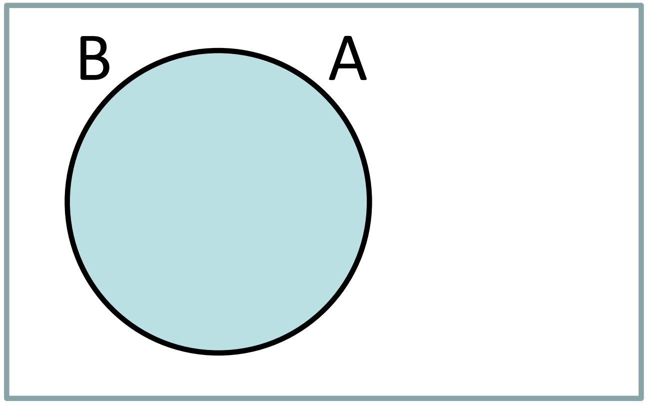


Useful theorems: Implication

Implication: If A logically entails B, then $P(B) \geq P(A)$.



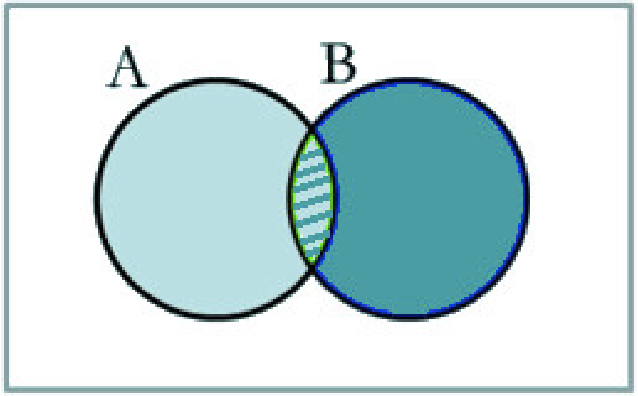
NB: In this diagram $P(B) > P(A)$.



NB: In this diagram $P(B) = P(A)$.

Useful theorems: General and special addition

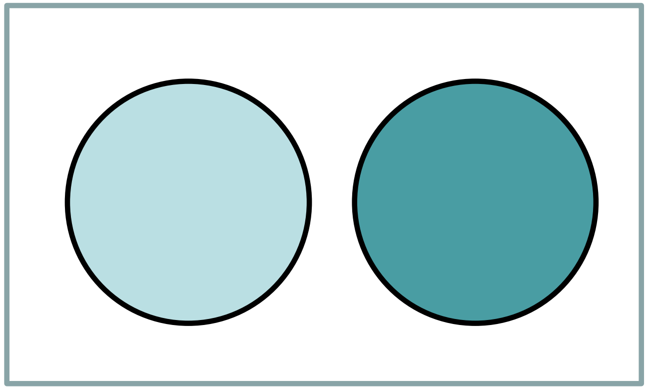
General addition: $P(A \vee B) = P(A) + P(B) - P(A \,\&\, B)$



NB: In this diagram $P(A \,\&\, B) \neq 0$, i.e. A, B are not mutually exclusive.

We subtract $P(A \,\&\, B)$ as we only want to count that area once.

Special addition: $P(A \vee B) = P(A) + P(B)$.

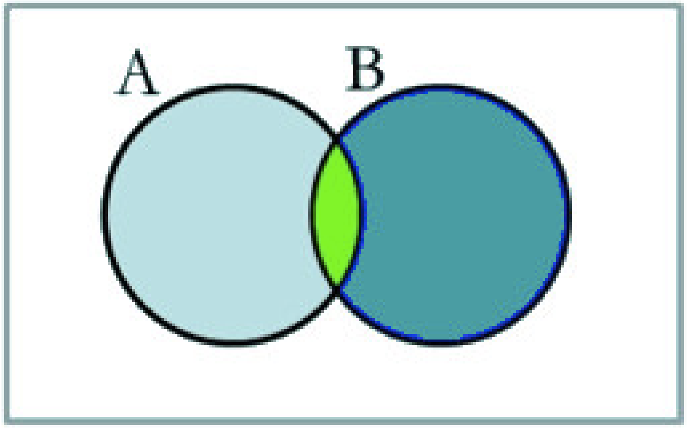


NB: In this diagram $P(A \,\&\, B) = 0$, i.e. A, B are mutually exclusive.

The last term of general addition rule can thus be omitted.

Conditional probabilities

- $P(A|B)$ is read as 'the probability of A, given B (is true)'.



non-exclusive case

- **Definition**:
  $P(A|B) = P(A \& B) / P(B)$
  where $P(B) > 0$.

  NB: In some axiomatisations such probabilities are primitive.

- The importance of conditional probabilities becomes clear when it is recognised that we can ask questions like:

  * $P(T_1|E_1) = ?$
  * $P(E_1|T_1) = ?$

Bayesian Confirmation Theory

Bayes theorem

- From the probability axioms one can derive Bayes Theorem:

$$P(H|E) = \frac{P(E|H) \times P(H)}{P(E)}$$

where $P(E) > 0$.

$P(H|E)$ is the posterior probability of the hypothesis.
$P(E|H)$ is the likelihood of the evidence.
$P(H)$ is the prior probability of the hypothesis.
$P(E)$ is the prior probability of the evidence.

Bayesianism, conditionalisation and relevance

- Bayes theorem on its own is not an account of confirmation.
- **Bayesianism** is such an account. It takes Bayes theorem as a starting point and adds the following epistemic principle:

  The (simple) principle of conditionalisation (a.k.a. the 'updating rule'): $P_{\text{new}}(H) = P_{\text{initial}}(H|E) = p$

- That means the new prior probability of *H* is updated to be the old posterior probability of *H* given *E*.
- **The relevance criterion of confirmation:**
  * *E* confirms *H* if and only if $P(H|E) > P(H)$
  * *E* disconfirms *H* if and only if $P(H|E) < P(H)$
  * *E* is neutral to *H* if and only if $P(H|E) = P(H)$

Confirmation measures

- To turn Bayesianism into a full-blown quantitative account of confirmation we need a **measure**.

- That is, we need a way to quantify the degree of confirmation a piece of evidence lends to a hypothesis.

- Several have been proposed (see Crupi et al 2007):

TABLE 1. ALTERNATIVE BAYESIAN MEASURES OF CONFIRMATION.

$D(e, h) = p(h \mid e) - p(h)$	Carnap ([1950] 1962)
$S(e, h) = p(h \mid e) - p(h \mid \neg e)$	Christensen (1999)
$M(e, h) = p(e \mid h) - p(e)$	Mortimer (1988)
$N(e, h) = p(e \mid h) - p(e \mid \neg h)$	Nozick (1981)
$C(e, h) = p(e \,\&\, h) - p(e) \cdot p(h)$	Carnap ([1950] 1962)
$R(e, h) = [p(h \mid e)/p(h)] - 1$	Finch (1960)
$G(e, h) = 1 - [p(\neg h \mid e)/p(\neg h)]$	Rips (2001)
$L(e, h) = [p(e \mid h) - p(e \mid \neg h)]/[p(e \mid h) + p(e \mid \neg h)]$	Kemeny and Oppenheim (1952)

Applications

- Bayesianism has numerous applications. For example, there are Bayesianist branches of the following theories:

  Confirmation theory – Seeks to analyse relations of support between evidence and theory.

  Formal learning theory: Mathematical treatment of how an agent learns and in particular how they ought to learn.

  Decision theory – Mathematical treatment of (rational) behaviour and choice-making in the presence of options.

Eliciting degrees of belief

- Suppose probabilities express an agent's (rational) degrees of belief in propositions, events, etc.

- How do we elicit such degrees of belief? One suggestion is through a *willingness to bet*!

  If I have a high confidence that *A* is true, then I would presumably be very willing to bet on it.

  My degree of belief should be high: $P(A) \approx 1$.

- Some, Jeffrey (1983), object to this betting approach:

  Irrational to bet that 'everyone dies next year'. If I win I can't collect. So cannot always elicit degrees of belief via betting.

How to choose priors: Two schools of thought

- How do we assign values to *P*(H) and *P*(E)? Probability theory provides few constraints, e.g. values from 0 to 1.

- There are two schools of thought within Bayesianism.

(1) **Subjective Bayesians**: Over and above those constraints, our choice of priors is a subjective matter.

<u>Supporters</u>: De Finetti, Howson, Jeffrey and Ramsey.

(2) **Objective Bayesians**: Choice of priors can be further constrained, e.g. the principle of indifference.

<u>Supporters</u>: Jaynes, Rosenkrantz and Williamson.

Dutch-book arguments

- **Dutch-book:** A combination of wagers that guarantees a loss through the violation of the axioms of probability.

- Offered not only to justify Bayesianism (+ conditionalisation principle) but also the axioms of probability.

  Example (synchronic case):
  Suppose person *K* is willing to pay 51p for a wager that returns £1 and 50p for the contrary wage that returns £1.

  Taken together the two bets guarantee a 1p loss for *K* (and a 1p gain for the bookie). In probabilistic terms:

  K assigns $P(W)$=0.51 and $P(\neg W)$=0.50. Recall that $P(A \vee \neg \mathrm{A})$=1. *K* violates this theorem (and axiom) since $P(W \vee \neg W)$> 1.

Some Positives

Advantage: Confirmation by entailment

- Bayesianism can account for **confirmation by entailment** and thus is meant to do justice to the H-D account.

  Suppose: $H \vdash E$. That means: $P(E|H) = 1$

  This simplifies our equation to: $P(H|E) = P(H) / P(E)$.

- Recall that if $H \vdash E$ then $P(E) \geq P(H)$. In short, it could not be that $P(E) < P(H)$. This would violate the first axiom.

- Either $P(E) = P(H)$ in which case $P(H|E) = 1$. Or $P(E) > P(H)$ in which case $P(H|E) > P(H)$. In both cases we get confirmation.

Advantage: Unexpected evidence

- It can account for the *intuition* that **unexpected evidence has a higher confirmational value** than expected evidence.

- Proposal 1**:** The more surprising, the lower its probability, i.e. $P(E)$ is close to zero.

- That's actually not enough to guarantee $P(H|E) \gg P(H)$. Suppose $P(H)$=0.5, $P(E|H)$=0.1, $P(E)$=0.1. Then $P(H/E)$=0.5.

- Proposal 2: How about requiring that $P(E|H)>P(E)$?

- Here H gets confirmed but again no guarantee $P(H|E) \gg P(H)$! Suppose $P(H)$=0.5, $P(E|H)$=0.11, $P(E)$=0.1. Then $P(H/E)$=0.55.

Advantage: Unexpected evidence (2)

- What we need is $P(E|H) \gg P(E)$!
- This accords well with Popper's demand that theories entail statements that are risky, i.e. risky predictions.

Suppose $H \vdash E_1, E_2$; $P(E_1|H) = 1$; $P(E_2|H) = 1$; $P(E_1) = .9$; $P(E_2) = .51$

$P(H|E_1) = \mathbf{1} * \mathbf{0.5} / \mathbf{0.9} = 0.5555...$
$P(H|E_2) = \mathbf{1} * \mathbf{0.5} / \mathbf{0.51} = 0.9803...$

Advantage: Raven paradox solution

- Bayesianism can provide a **solution to the raven paradox**.
- *H*: All ravens are black, E_1: *a* is black raven, E_2: *b* is a white sock.

  It seems reasonable to assert, especially if *a* is amongst the first black ravens that we observe, that $P(E_1|H)$ » $P(E_1)$.

  This is not the case with non-black non-ravens since almost everything is such a thing. So, $P(E_2|H) \approx P(E_2)$.

  So, $P(E_1|H)/P(E_1)$ » $P(E_2|H)/P(E_2)$. Hence E_2 confirms *H* but much less than E_1.

  Rationale: If the world has finite # of things, the more we eliminate as non-black non-ravens, the more confidence in *H*.

Some Negatives

The choice of priors

- *Subjective Bayesians*: We are free to choose degrees of belief in a given proposition so long as they are coherent.

- **Problem**: People can assign wildly divergent priors to one and the same hypothesis. How can we confirm anything?

- **Replies:**

  (1) Avoid such extreme and implausible priors.

  (2) Unless they start like that, their posteriors should (eventually) converge as the evidence accumulates.

  <u>NB</u>: Called the 'washing out' 'swamping out' of the priors.

The old evidence problem

- Glymour (1980): Oftentimes scientists appeal to evidence that's already known to support their theories.

- If evidence is 'old', then we might say $P(E) = 1$. That reduces our equation to: $P(H|E) = P(E|H) * \mathrm{P}(H)$.

  If $P(E|H) < 1$, then $P(H|E) < P(H)$ [*disconfirmation*]
  If $P(E|H) = 1$, then $P(H|E) = P(H)$ [*no confirmation*].

- Historical example: Mercury's perihelion was discovered in the 19th c. but confirmed Einstein's GTR in early 20th c.

- **Reply:** Agent must evaluate *H* counterfactually (holding all the same beliefs minus *E*). Thus $P(E) < 1$. See Howson (1991).

Special Topic:
Novel Predictions

Introduction

- *Novel predictions*, *novel facts* or *novel evidence* are in some respect phenomena that are 'unexpected' vis-à-vis a theory.

  NB: This sense is explicated below.

- There is a long history of treating such unexpected phenomena as possessing special confirmational weight.

  * Poisson's spot
  * Starlight delfection
  * Mercury's perihelion
  * …

Accomodationism vs. predictivism

- Two families of views regarding their confirmational value.

Accommodationism: Accommodated evidence carries as much confirmational weight as predicted evidence.

Predictivism: Accommodated evidence (or even evidence that *could have been accommodated*) carries less/no weight.

Predictivism: Two families

- **Temporal-novelty** (e.g. Maher 1988): Phenomena known after *H* was formed have more (/the only) evidential weight.

- **Use-novelty** (e.g. Worrall 2002): Phenomena not used in the construction of *H* have more (/the only) evidential weight.

  Rationale: A hypothesis could not possibly have been shaped to accommodate phenomena that were:

  * unknown prior to its formulation
  * not used in its construction

- This rules out that the hypothesis is ad hocly constructed.

The End